BOSTON UNIVERSITY

EK 307: Electric Circuits

Fall 2017

Lecture 17

Nov 7, 2017

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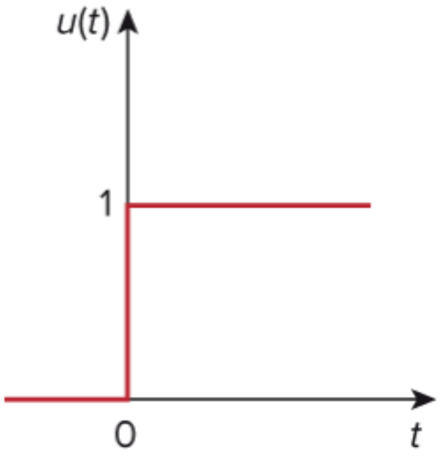
Last time (Lecture 16):

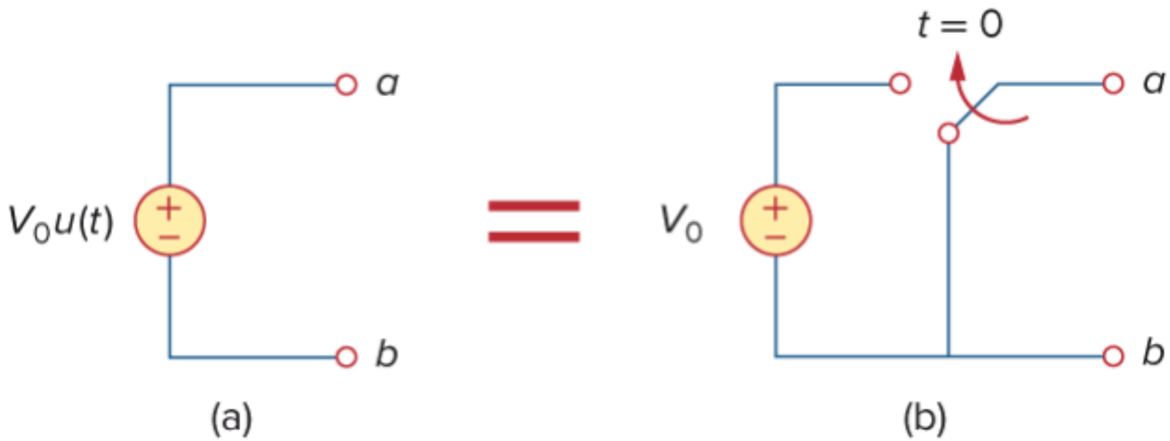
- **First-order circuits**
- **Source-free RC and RL circuits**
- **Singularity functions to represent step responses**
- **Discussed a driven RC circuit**

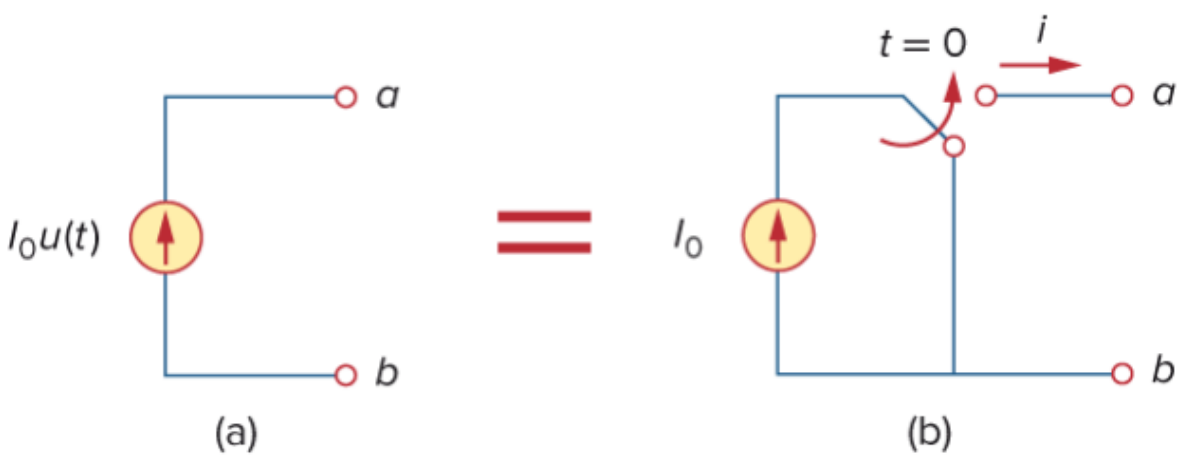
(covered Sections 7.1-7.4 and a little of 7.5)

Singularity functions: unit step, delta, ramp

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$




t = 0
a
$V_0u(t)$
V_0
b
(a)
(b)


t = 0
i
a
$I_0u(t)$
I_0
b
(a)
(b)

Lecture 17: Chapter 7: 1^{st} order ccts

1. **Driven RC and RL circuits**
2. **First-order op-amp circuits**

(covering sections 7.5-7.7)

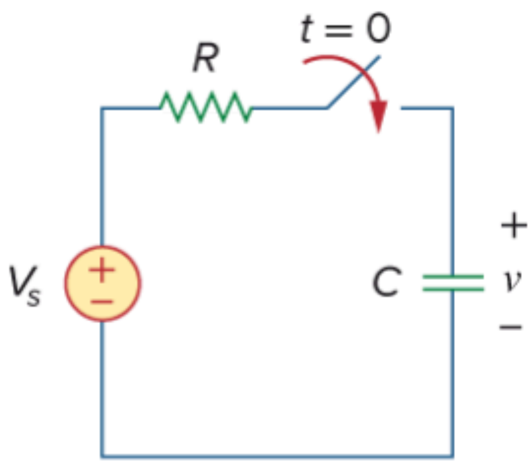
Step response

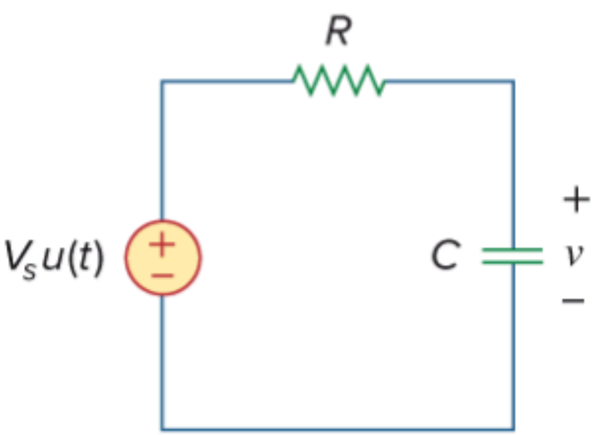
The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

From last time: discussed physical intuition

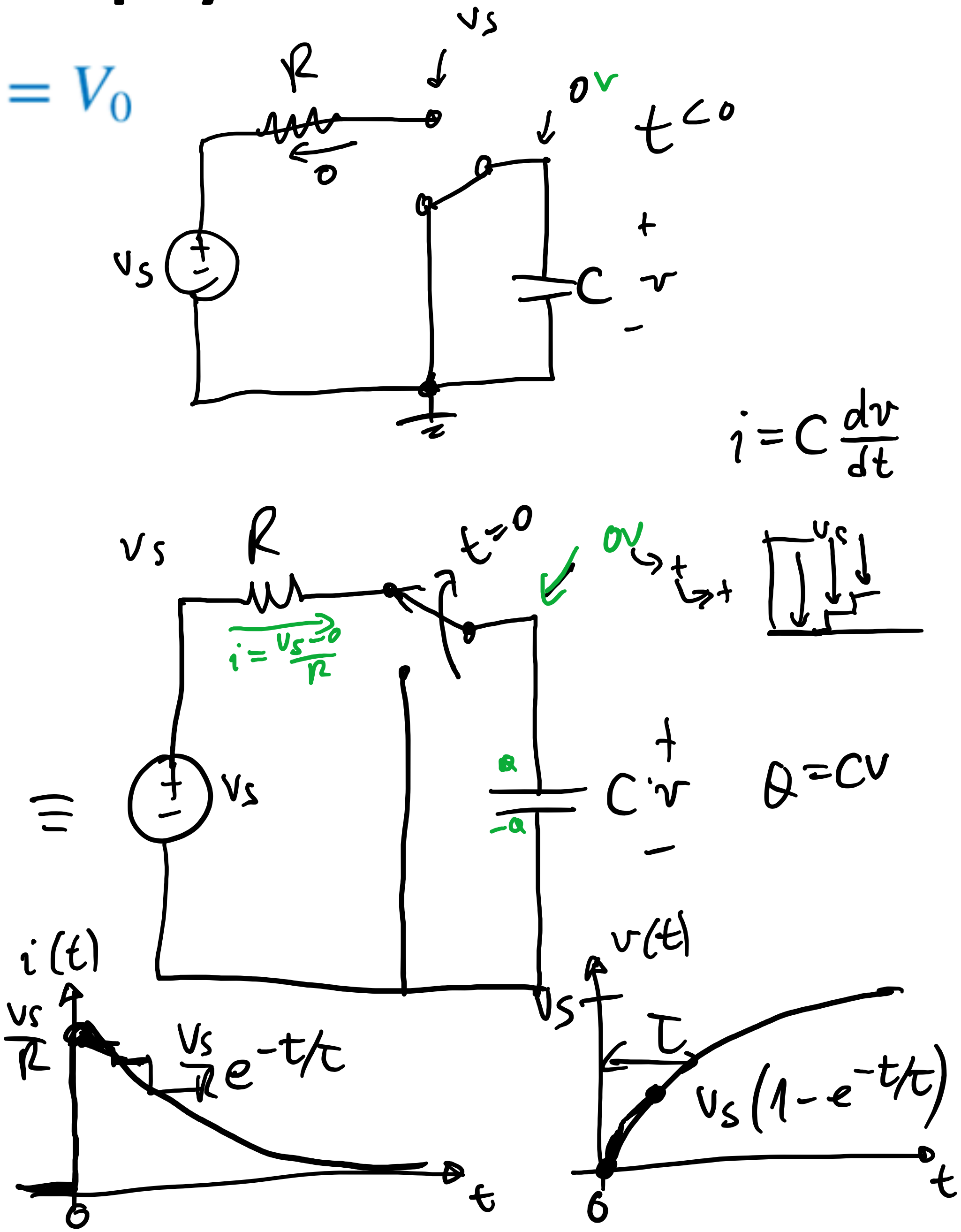
$$v(0^-) = v(0^+) = V_0$$

Representation using switch:

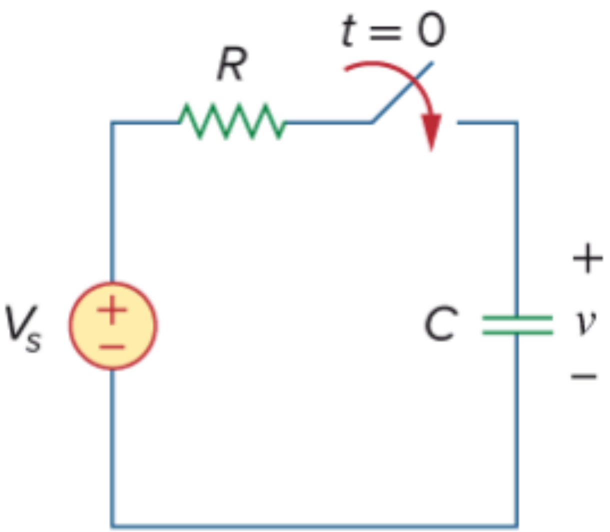




Representation using unit step function



Solving for the step response



We write Kirchhoff's current law

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

where v is the voltage across the capacitor. For $t > 0$, **Eq. (7.41)** becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

or

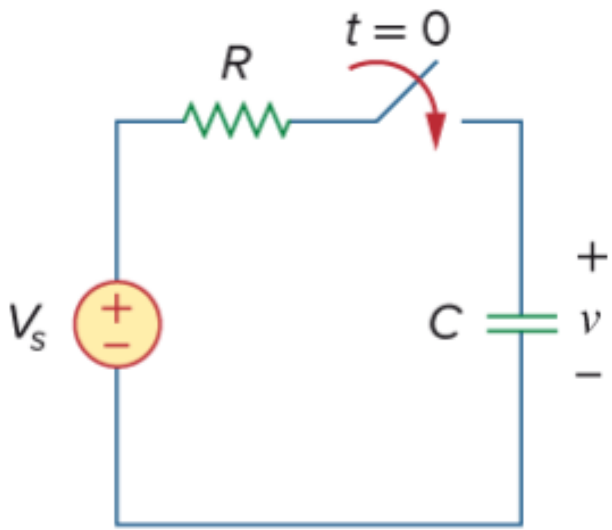
$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

Solving for the step response



$$\ln \frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

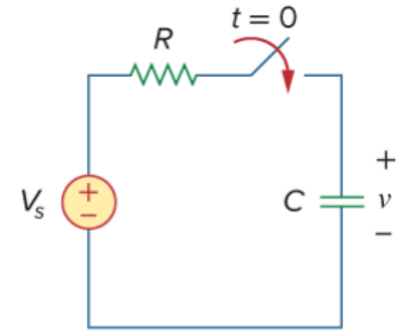
$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

Thus,

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

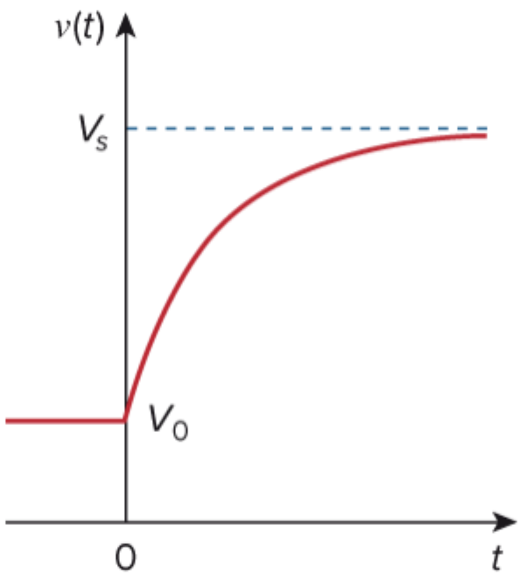
This is known as the **<u>complete (or total) response</u>** of circuit to sudden turn-on of DC voltage source, assuming capacitor is initially charged.

Solving for the step response

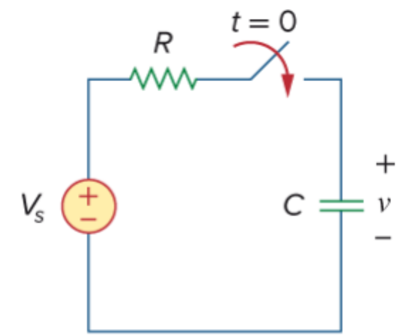


$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

Assuming that $V_s > V_0$:



Solving for the step response



If capacitor is uncharged initially, then $V_0 = 0$ and

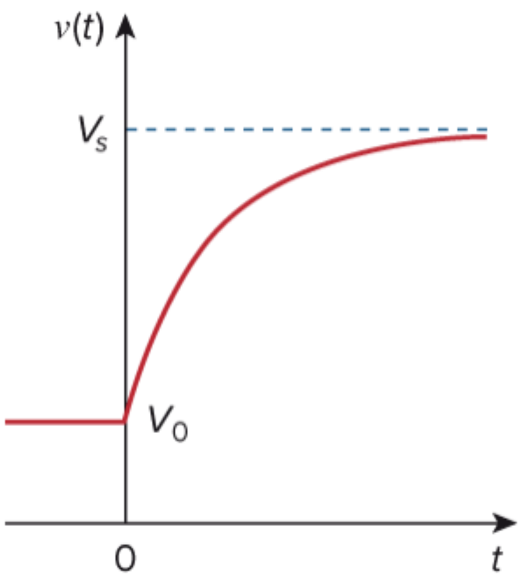
$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

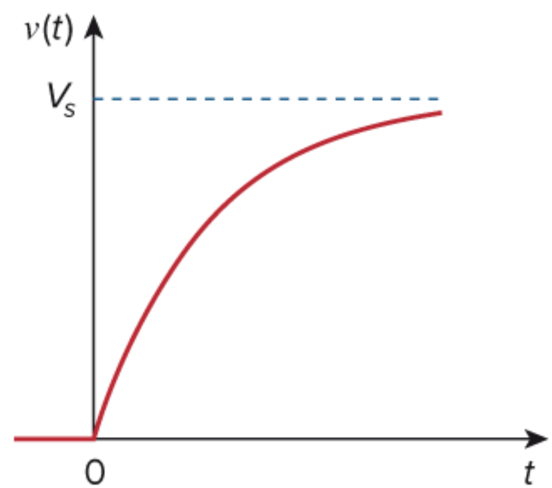
$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

or

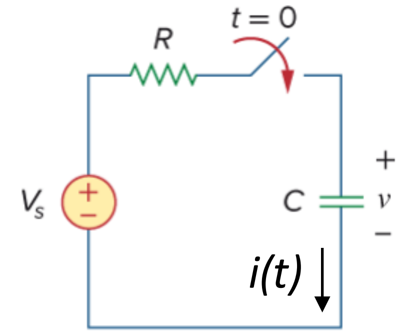
$$v(t) = V_s(1 - e^{-t/\tau})u(t)$$

Assuming that $V_s > V_0$:





Solving for the step response

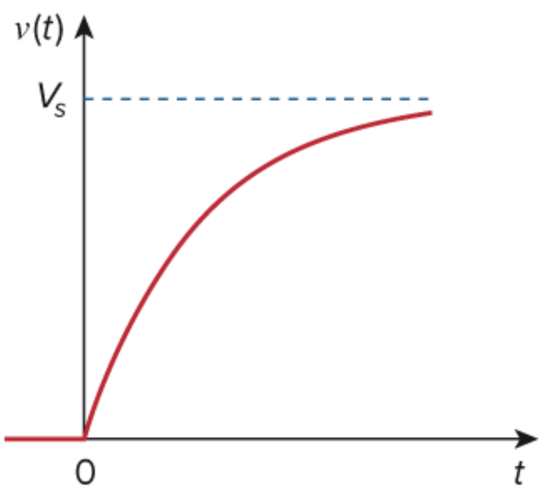


If capacitor uncharged initially, then $V_0 = 0$ and

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

or

$$v(t) = V_s(1 - e^{-t/\tau})u(t)$$



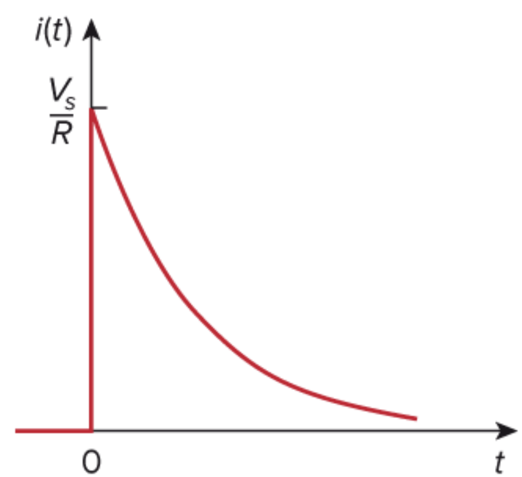
Voltage response

Current:

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad \tau = RC, \qquad t > 0$$

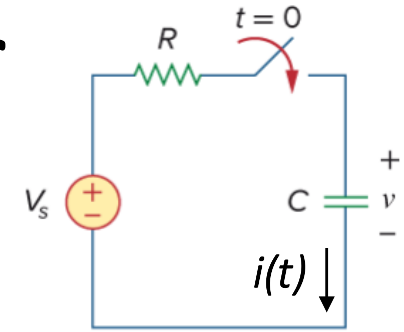
or

$$i(t) = \frac{V_s}{R}e^{-t/\tau}\, u(t)$$



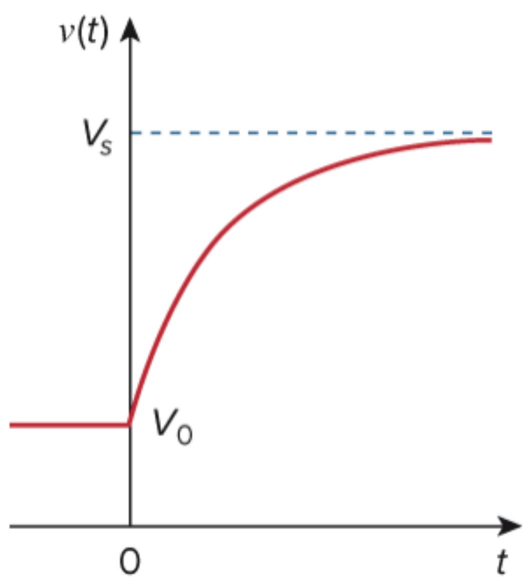
Current response

Systematic shortcut method for step response of RC/RL circuit



$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

Picture 1: Break down into natural and forced response



Voltage response

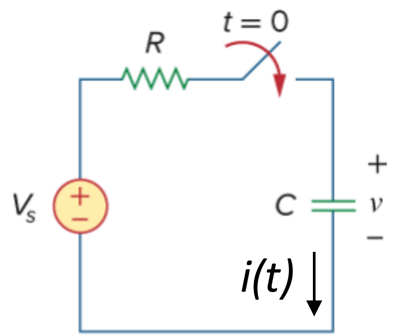
Complete response = natural response (stored energy) + forced response (independent source)

$$v = v_n + v_f$$

$$v_n = V_o e^{-t/\tau}$$

$$v_f = V_s(1 - e^{-t/\tau})$$

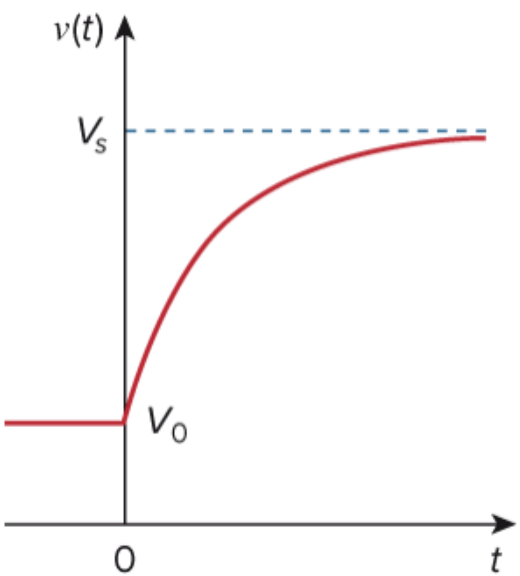
Systematic shortcut method for step response of RC/RL circuit



$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

Picture 2: Break down into **transient** + **steady state** responses

$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}$$

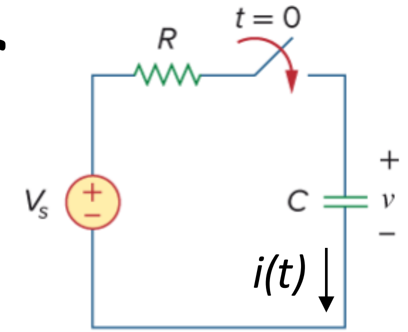


Voltage response

The **transient response** is the circuit's temporary response that will die out with time.

The **steady-state response** is the behavior of the circuit a long time after an external excitation is applied.

Systematic shortcut method for step response of RC/RL circuit



$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)\, e^{-t/\tau}, & t > 0 \end{cases}$$

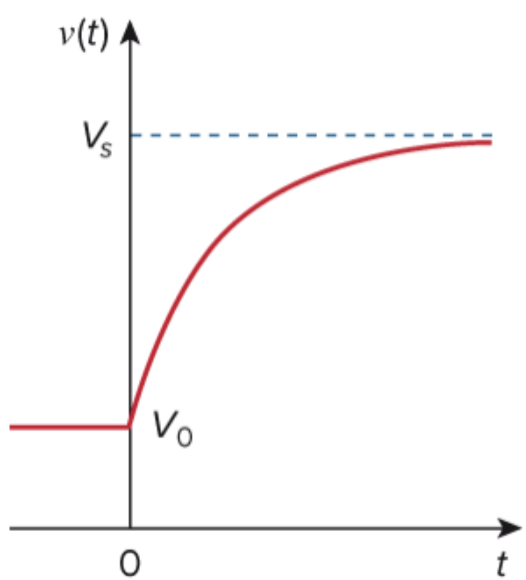
Picture 2: Break down into **transient** + **steady state** responses

Complete response = transient response (temporary part) + steady-state response (permanent part)

$$v = v_t + v_{ss}$$

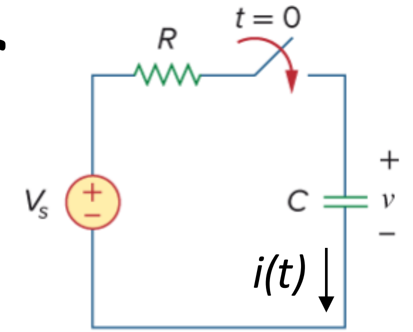
$$v_t = (V_o - V_s)e^{-t/\tau}$$

$$v_{ss} = V_s$$



Voltage response

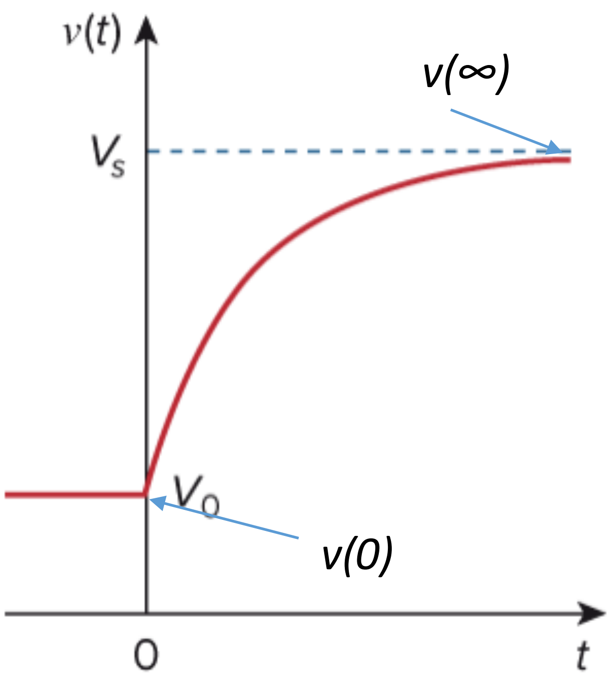
Systematic shortcut method for step response of RC/RL circuit



Either way, can write response for t>0 as:

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

(For the capacitor voltage!)



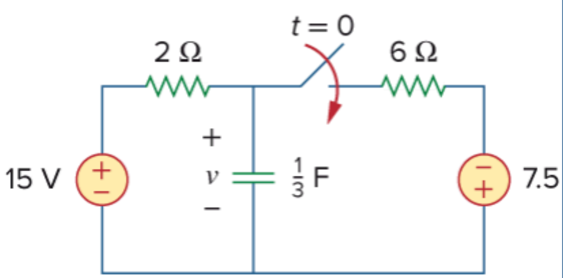
To find step response of *RC* circuit need 3 things:

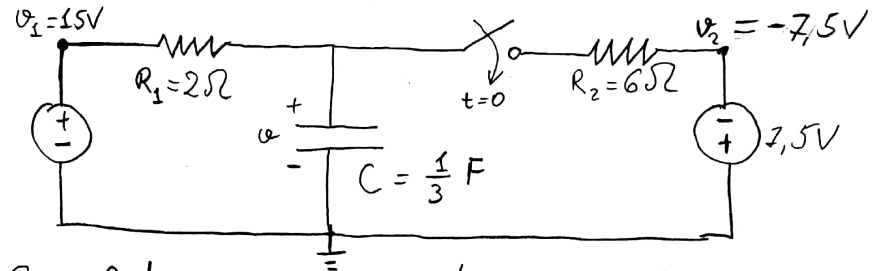
1. Initial capacitor voltage, *v(0)*
2. Final capacitor voltage, *v(∞)*
3. Time constant, τ

(Get #1 from cct at t<0, and #2,#3 from cct at t>0)

Practice Problem 7.10

Find $v(t)$ for $t > 0$ in the circuit of Fig. 7.44. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$.





Complete response = transient response + steady-state resp.

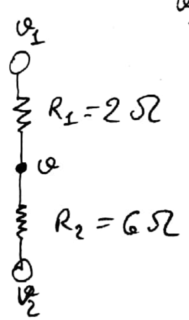
⇓

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}}$$

at $t = 0^-$ the switch is open. For dc voltage capacitor is open circuit, therefore

$$v(0) = v_1 = 15V$$

at $t = \infty$ voltage on the capacitor is determined by voltage divider.



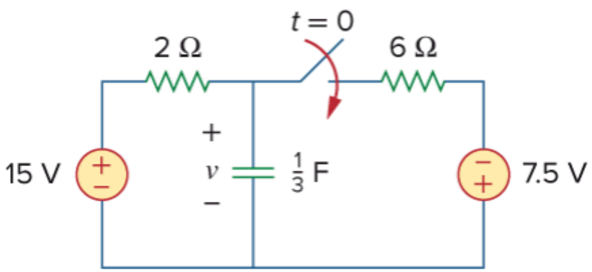
$$v(\infty) = v_2 + \frac{v_1 - v_2}{R_1 + R_2} =$$

$$= -7,5V + \frac{22,5V}{8\Omega} \cdot 6\Omega =$$

$$= -7,5V + 16,875V = 9,375V$$

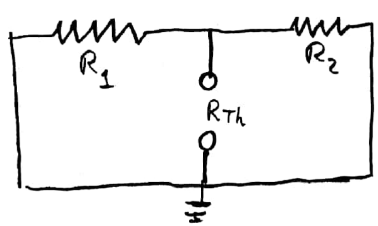
Practice Problem 7.10

Find $v(t)$ for $t > 0$ in the circuit of Fig. 7.44. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$.



$\underline{\tau}$ To find τ we need to find resistance seen by the capacitor i.e. Thevenin resistance.

Note, that output resistance of voltage soure is 0.



$R_T = \frac{R_1 R_2}{R_1 + R_2}$ $\tau = R_{Th} \cdot C = \frac{6 \cdot 2}{8} \cdot \frac{1}{3} s = \frac{1}{2} s$

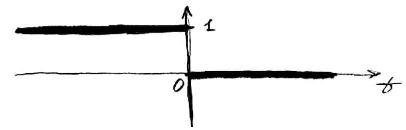
$\underline{v(t)}$

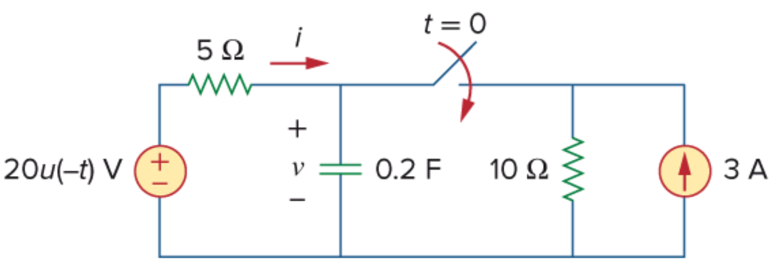
$v(t) = 9{,}375 V + (15 - 9{,}375) e^{-2t} V = (9{,}375 + 5{,}625 e^{-2t}) V$

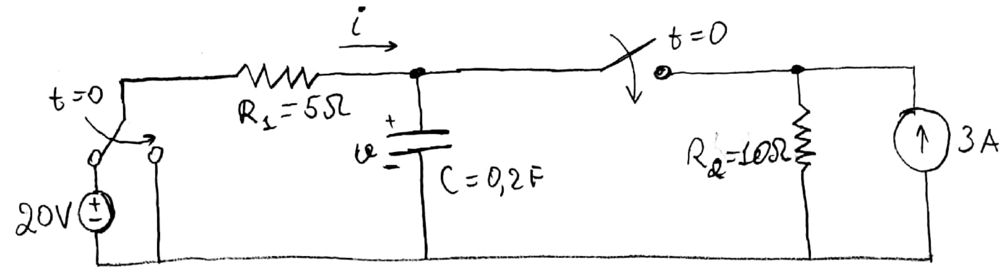
$v(0{,}5) = (9{,}375 + 2{,}0693) V = 11{,}44 V$

Practice Problem 7.11

The switch in Fig. 7.47 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

$u(-t) =$ 



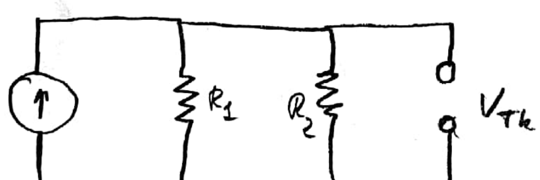


$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}}$$

$t < 0$ For DC signal capacitor has infinite resistance.

$\Rightarrow \quad i(t) = 0 \quad \Rightarrow v(t) = 20V, \; t < 0 \Rightarrow v(0) = 20V$

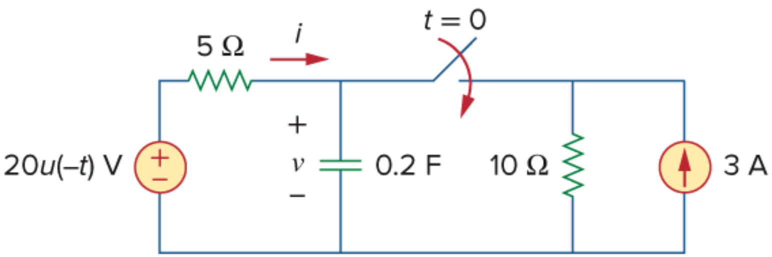
$t = \infty$ Infinitely long time after switches are turned, voltage on the capacitor can be found from equivalent circuit

$I = 3A$ 

$$v(\infty) = \frac{R_1 R_2}{R_1 + R_2} \cdot I = \frac{50}{15} \cdot 3\,V = 10V$$

Practice Problem 7.11

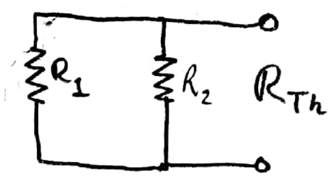
The switch in Fig. 7.47 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.



τ

Output risistance of a current source is $\infty \Rightarrow$

$\Rightarrow$



$$R_{Th} = \frac{R_1 R_2}{R_1 + R_2}$$

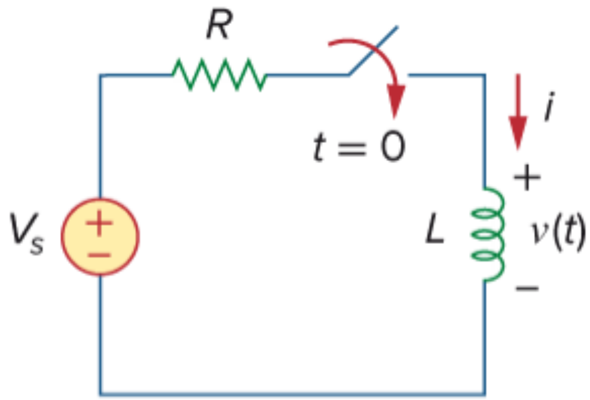
$$\tau = R_{Th} \cdot C = \frac{50}{15} \cdot \frac{1}{5}\, s = \frac{10}{15}\, s$$

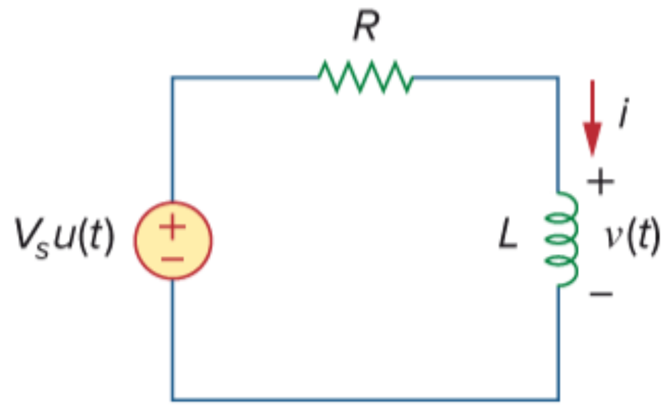
$v(t), i(t)$

$t > 0 \quad v(t) = 10\,V + (20 - 10)e^{-1{,}5t}\,V = 10(1 + e^{-1{,}5t})\,V$

$t > 0 \quad i(t) = -\frac{v(t)}{R_1} = -2(1 + e^{-1{,}5t})\,A$

Step Response of RL Circuits

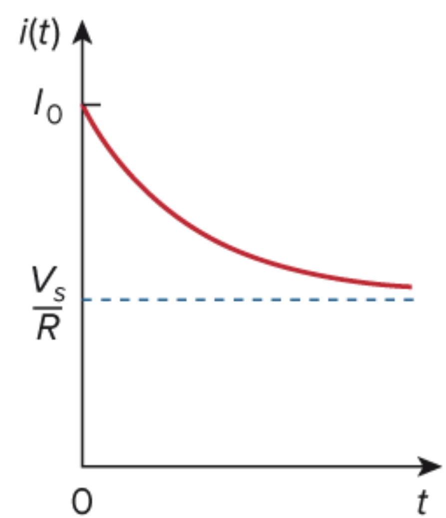




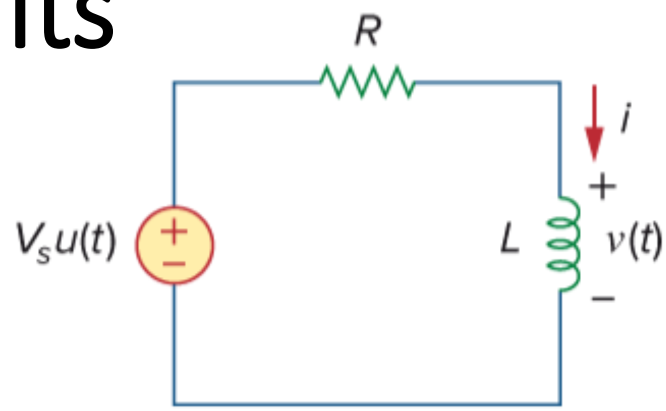
Same as RC, but here response is for **<u>inductor current</u>** (it's always the variable that stores energy – ½Cv² for cap, ½Li² for inductors)

$$i(t) = i(\infty) + [\,i(0) - i(\infty)\,]e^{-t/\tau}$$

$$\tau = \frac{L}{R}$$

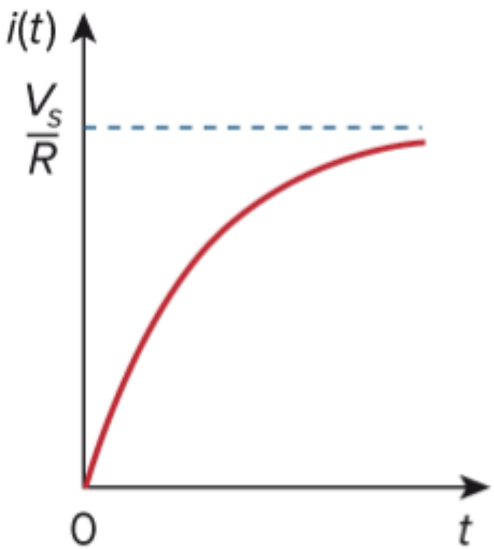


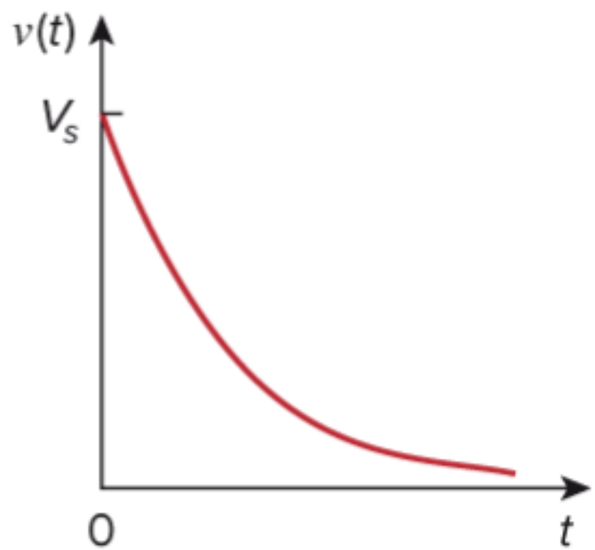
Step Response of RL Circuits



$$i(t) = i(\infty) + [\, i(0) - i(\infty)\,]e^{-t/\tau}$$

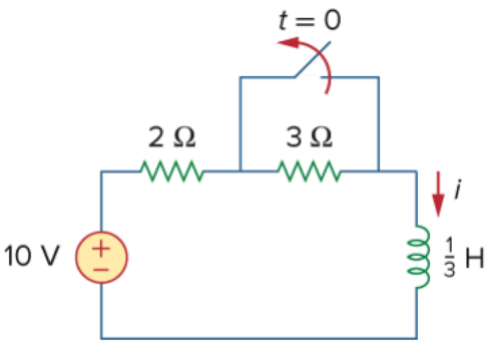
$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

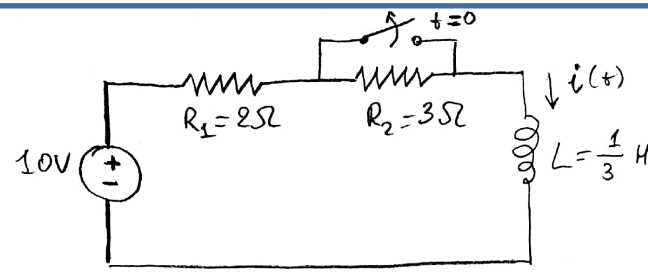




Example 7.12

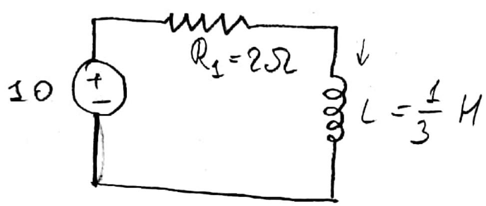
Find $i(t)$ in the circuit of **Fig. 7.51** for $t > 0$. Assume that the switch has been closed for a long time.





$t>0$ $\quad i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau}}$

$\underline{t<0}$ equivalent circuit



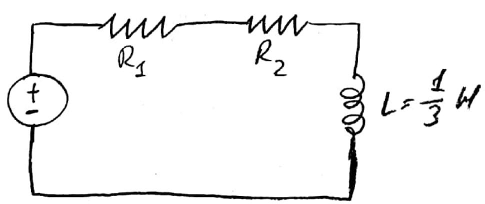
For DC current inductor is a short circuit ⇒

$\Rightarrow i(t) = \frac{10V}{2\Omega} = 5A, \quad t<0$

$\Downarrow$

$i(0) = 5A$

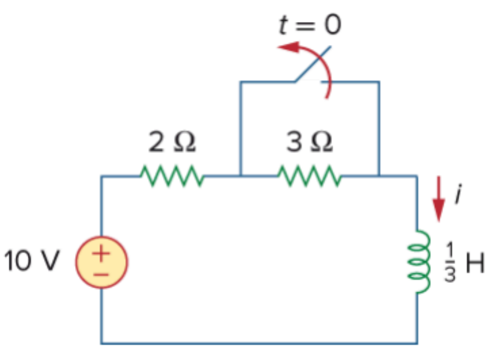
$\underline{t = \infty}$



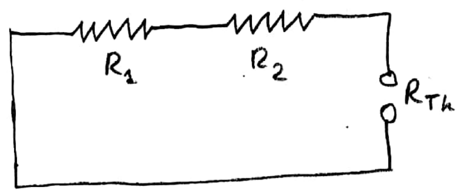
$i(\infty) = \frac{10}{R_1 + R_2} = 2A$

Example 7.12

Find $i(t)$ in the circuit of Fig. 7.51 for $t > 0$. Assume that the switch has been closed for a long time.



τ



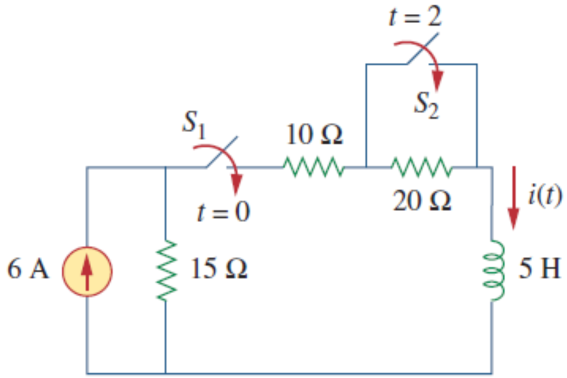
$$\tau = \frac{L}{R_{Th}} = \frac{L}{R_1 + R_2} = \frac{1}{3} \cdot \frac{1}{5}\,s = \frac{1}{15}\,s$$

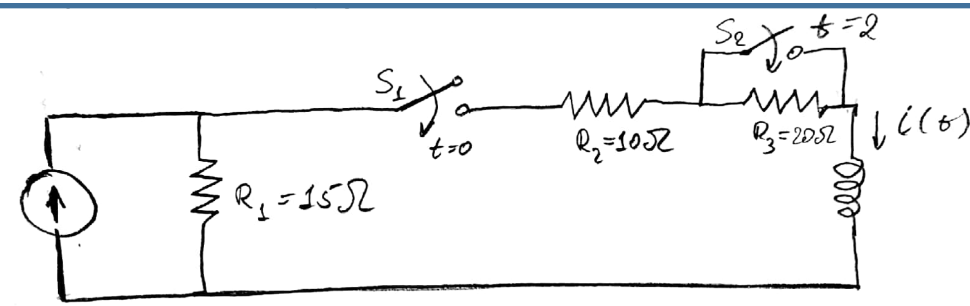
i(t)

$$i(t) = 2A + (5A - 2A)e^{-15t} = (2 + 3e^{-15t})A$$

Practice Problem 7.13

Switch S_1 in Fig. 7.54 is closed at $t = 0$, and switch S_2 is closed at $t = 2$ s. Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$.

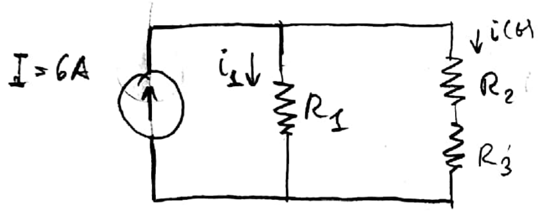




t<0 All the current flows through R_1, therefore

$i(t) = 0, \; t<0 \Rightarrow i(0) = 0$

0<t<2 equivalent circuit at $t \to \infty$, assuming S_2 switch stays open all the time.



$t \to \infty$

R_1 and series combination of R_2 and R_3 are connected in parallel, $\Rightarrow$

$\Rightarrow i_1 R_1 = i(\infty)(R_2 + R_3)$ also $i_1 + i(\infty) = I$

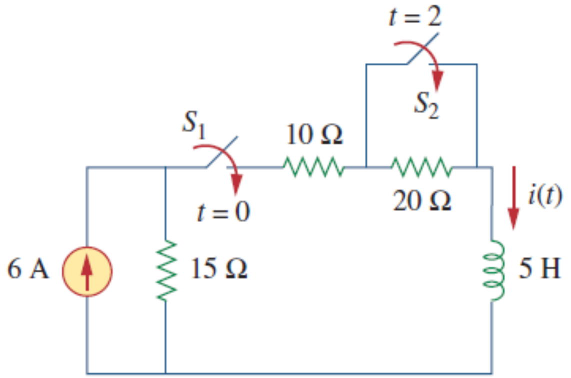
$\Downarrow$

$$[I - i(\infty)] R_1 = i(\infty)(R_2 + R_3)$$

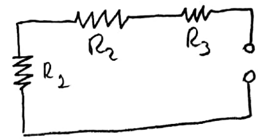
$$i(\infty) = \frac{I R_1}{R_1 + R_2 + R_3} = \frac{6 \cdot 15}{45} = 2A$$

Practice Problem 7.13

Switch S_1 in Fig. 7.54 is closed at $t = 0$, and switch S_2 is closed at $t = 2$ s. Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$.



τ_1



$$\tau_1 = \frac{L}{R_{Th}} = \frac{L}{R_1 + R_2 + R_3} = \frac{5}{45} s = \frac{1}{9} s$$

$i(t),\ 0<t<2$

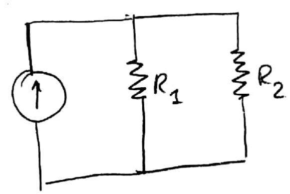
$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau}}$$

$$i(t) = 2A + (0-2)e^{-9t} A = 2(1 - e^{-9t}) A$$

$$t = 2s >> \tau_1 => i(2) = 2A$$

$t > 2$

$I = 6A$



$$i(\infty) = \frac{I R_1}{R_1 + R_2} = \frac{6 \cdot 15}{25} A = 3,6A$$

$$\tau_2 = \frac{L}{R_1 + R_2} = \frac{5}{25} s = \frac{1}{5} s$$

$i(t),\ t>2$

$$i(t) = i(\infty) + [i(2) - i(\infty)]e^{-5(t-2)} = 3,6 - 1,6e^{-5(t-2)}$$

complete response

$$i(t) = \begin{cases} 0, & t<0 \\ 2(1-e^{-9t})A, & 0<t<2 \\ 3,6 - 1,6e^{-5(t-2)}A, & t>2 \end{cases}$$

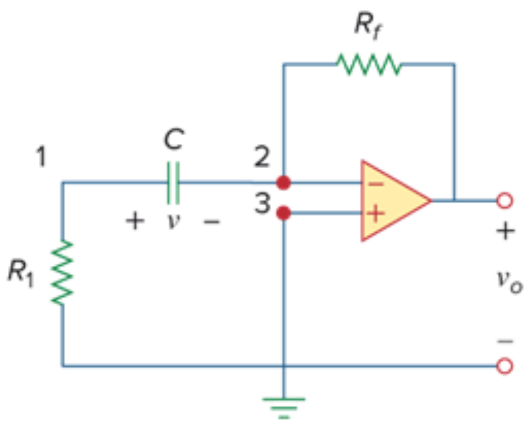
$i(1) = 1,9998A$ $\quad$ $i(3) = 3,583A$

First-order op-amp circuits

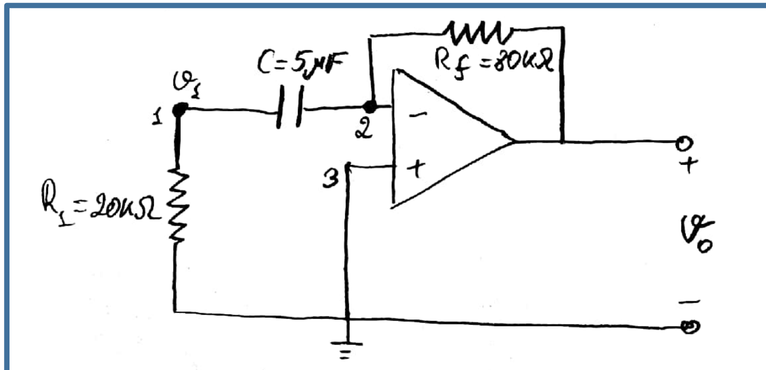
- Op-amp cct containing a storage element (C or L)
- Differentiators and integrators, already covered before, are examples of first-order op amp ccts
- For practical reasons, inductors are hardly ever used in op amp circuits because they are large.
- Therefore, the op amp circuits we consider here are of the *RC* type.

Example 7.14

For the op amp circuit in Fig. 7.55(a), find v_o for $t > 0$, given that $v(0) = 3$ V. Let $R_f = 80$ kΩ, $R_1 = 20$ kΩ, and $C = 5$ μF.



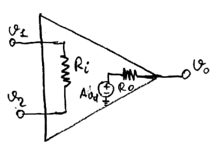
(a)



We write Kirchhoff's current law for point 1

$$\frac{0 - v_1}{R_1} = C\frac{dv}{dt}$$

op-amp equivalent circuit



ideal op-amp

$A = \infty$

$R_i = \infty\ \Omega$

$R_o = 0\ \Omega$

$i_1 = i_2 = 0$

$v_1 = v_2$

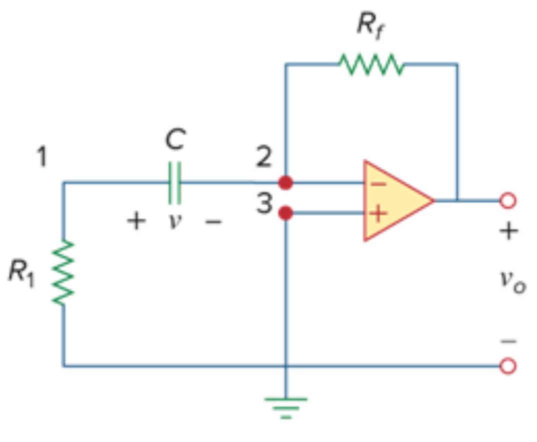
ideal op-amp $\Rightarrow$ $i_2 = i_3 = 0 \Rightarrow v_1 = v \Rightarrow$

$$\frac{dv}{dt} + \frac{v}{CR_1} = 0$$

$$v(t) = V_0 e^{-\frac{t}{\tau}} \qquad \tau = CR_1, \quad V_0 = v(0) = 3V$$

Example 7.14

For the op amp circuit in Fig. 7.55(a), find v_o for $t > 0$, given that $v(0) = 3$ V. Let $R_f = 80$ kΩ, $R_1 = 20$ kΩ, and $C = 5$ μF.



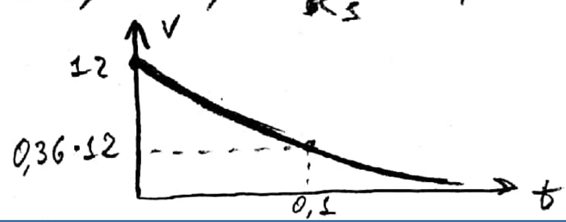
(a)

KCL for point 2

$$C\frac{dv}{dt} = \frac{0 - v_o}{R_f} \Rightarrow v_o(t) = -CR_f\frac{dv}{dt}$$

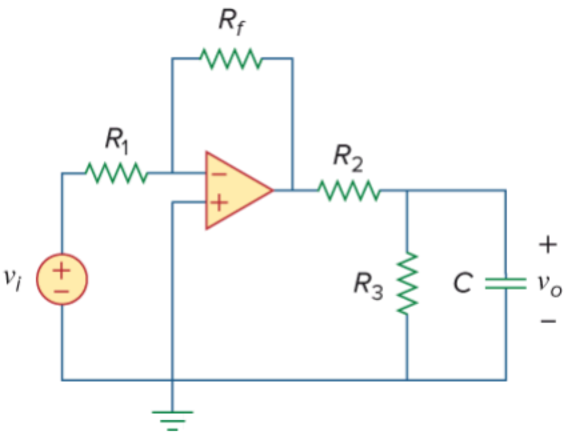
$$v_o(t) = \frac{CR_f}{CR_1} V_o e^{-\frac{t}{\tau}} = \frac{R_f}{R_1} V_o e^{-\frac{t}{\tau}}$$

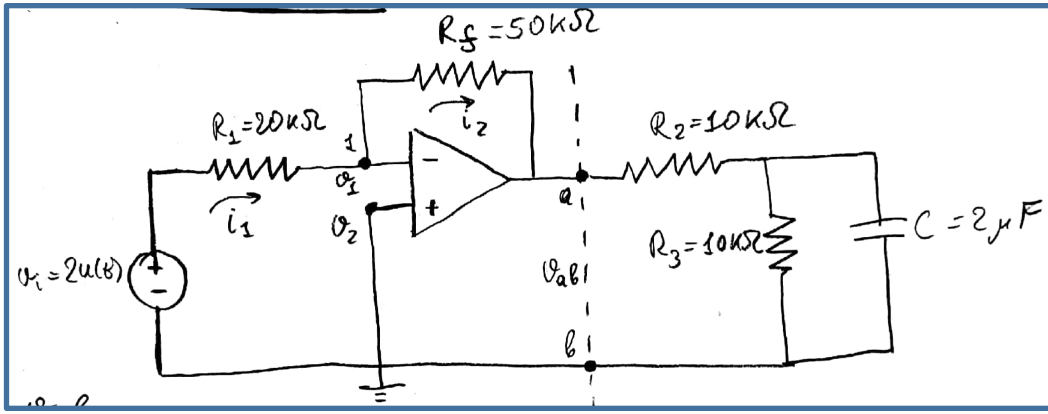
$$\tau = 5\cdot 10^{-6}\text{F} \cdot 20\cdot 10^{3}\Omega = 0{,}1\text{s}, \quad \frac{R_f}{R_1} = 4 \Rightarrow v_o(t) = 12\text{V}e^{-\frac{t}{0{,}1\text{s}}}$$



Example 7.16

Find the step response $v_o(t)$ for $t > 0$ in the op amp circuit of Fig. 7.59. Let $v_i = 2u(t)$ V, $R_1 = 20$ kΩ, $R_f = 50$ kΩ, $R_2 = R_3 = 10$ kΩ, $C = 2\ \mu$F.





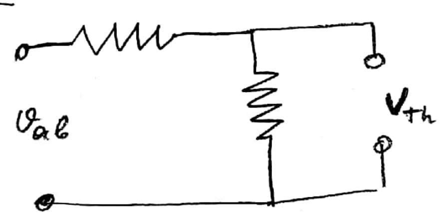
v_{ab}

KCL for point 1

$$i_1 = i_2 \Rightarrow \frac{v_i - v_1}{R_1} = \frac{v_1 - v_{ab}}{R_f}$$

$$v_1 = v_2 = 0 \Rightarrow \frac{v_i}{R_1} = -\frac{v_{ab}}{R_f} \Rightarrow v_{ab} = -\frac{R_f}{R_1} v_i$$

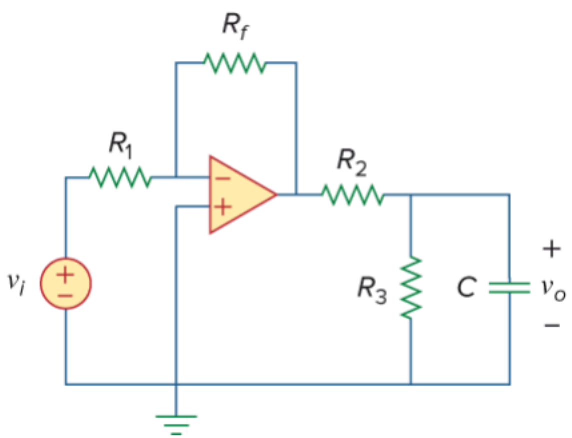
V_{Th}



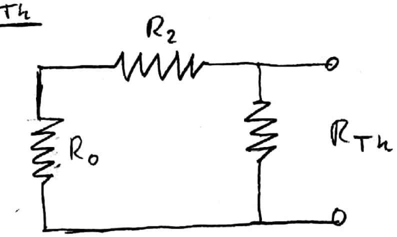
$$V_{Th} = \frac{R_3}{R_2 + R_3} v_{ab} = -\frac{R_3}{R_2 + R_3} v_i$$

Example 7.16

Find the step response $v_o(t)$ for $t > 0$ in the op amp circuit of **Fig. 7.59**. Let $v_i = 2u(t)$ V, $R_1 = 20$ kΩ, $R_f = 50$ kΩ, $R_2 = R_3 = 10$ kΩ, $C = 2$ μF.

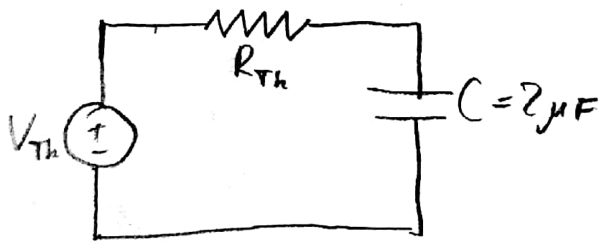


R_{Th}



$$R_{Th} = \frac{R_2 R_3}{R_2 + R_3}$$

equivalent circuit



$$V_{Th} = -\frac{10}{20} \cdot \frac{50}{20} \cdot 2u(t) = -2,5u(t)$$

$$R_{Th} = \frac{R_2 R_3}{R_2 + R_3} = \frac{10 \cdot 10}{20} k\Omega = 5 k\Omega$$

$$\tau = R_{Th} \cdot C = 5 \cdot 10^3 \Omega \cdot 2 \cdot 10^{-6} F = 0,01 s$$

$$v_o(t) = v_o(\infty) + [v_o(0) - v_o(\infty)] e^{-\frac{t}{0,01s}}, \quad t > 0$$

$$v_o(t) = -2,5 + [0 + 2,5] e^{-\frac{t}{0,01s}} = -2,5\left(1 - e^{-\frac{t}{0,01s}}\right) u(t)$$