BOSTON UNIVERSITY

EK 307: Electric Circuits

Fall 2017

Lecture 24

Dec 7, 2017

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Administrivia

- Course evaluation
- Homework material back
- Review/office hour – doodle poll

Today

- Chapter 11 (Sec 11.1 to 11.3):
  - Power in the phasor domain
  - Max power transfer

- Chapter 14:
  - Frequency response and intro to filter circuits

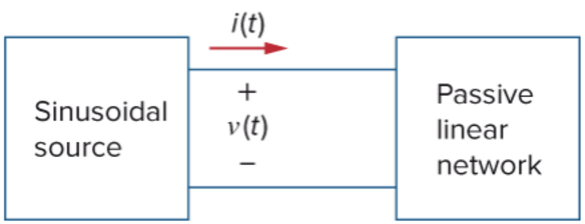
Chapter 11: Power in phasor domain

Instantaneous and average power

$$\boxed{p(t) = v(t)i(t)}$$

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

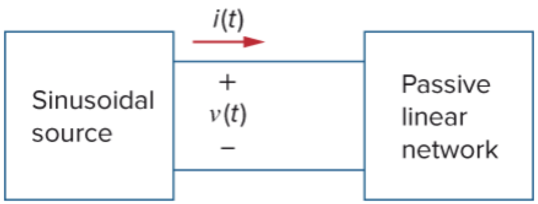


Instantaneous and average power

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

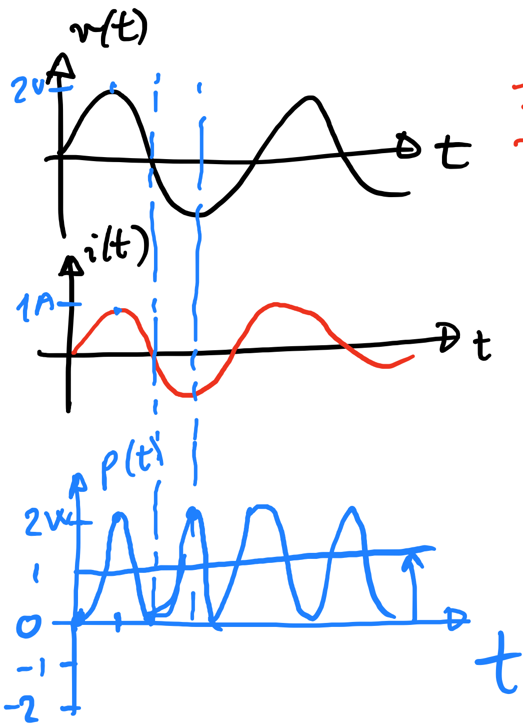
$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

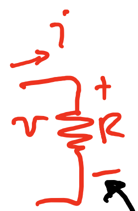


$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

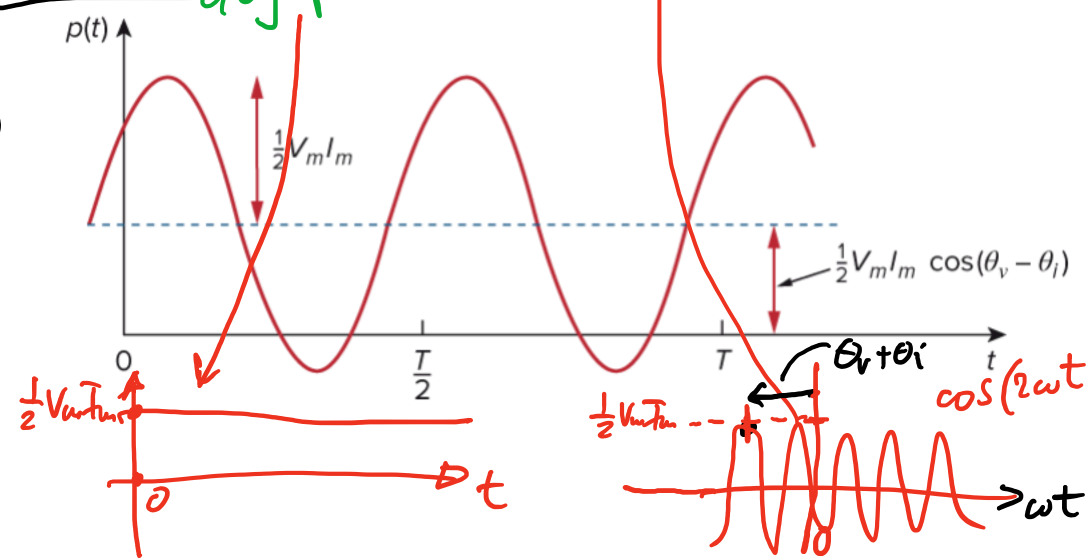
Instantaneous power:

$$\boxed{p(t) = \underbrace{\frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}_{\text{avg power}} + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)}$$





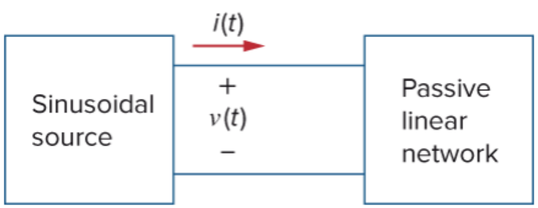
p(t) burned in resistor W (J/s)



Instantaneous and average power

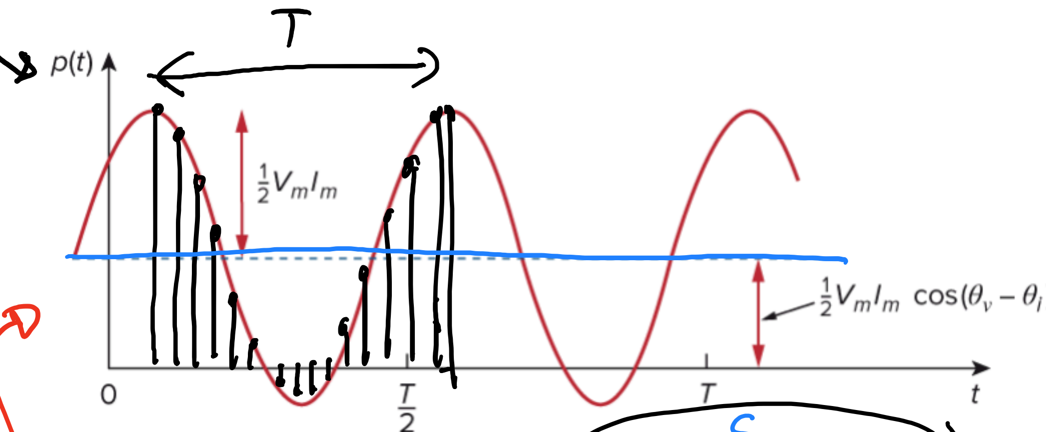
$$v(t) = V_m \cos(\omega t + \theta_v)$$

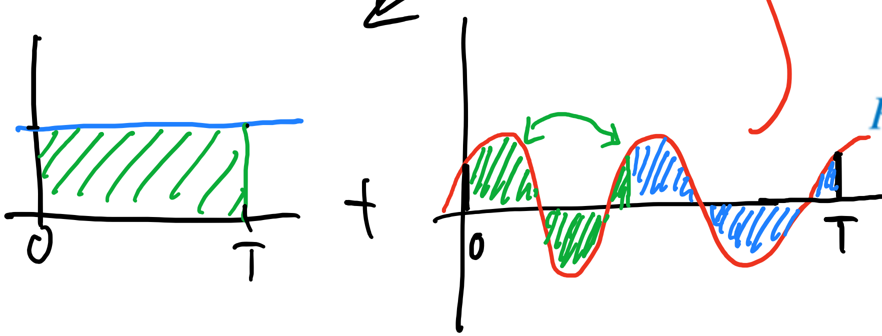
$$i(t) = I_m \cos(\omega t + \theta_i)$$



Instantaneous power:

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$



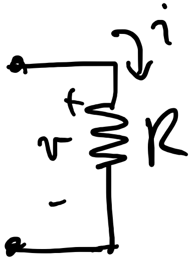


$$P = \frac{1}{T}\int_0^T p(t)\,dt = \frac{1}{T}\int_0^T \Big[\overset{C}{\boxed{\tfrac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}} + \tfrac{1}{2} V_m I_m \cos(2\omega t + \dots\Big]\,dt$$

$$= \frac{1}{T}\int_0^T C\,dt + \frac{1}{T}\int_0^T \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \cancel{\frac{1}{T}}\, C \cancel{T} + \cancel{\frac{1}{T}\int_0^T \dots} \to 0$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

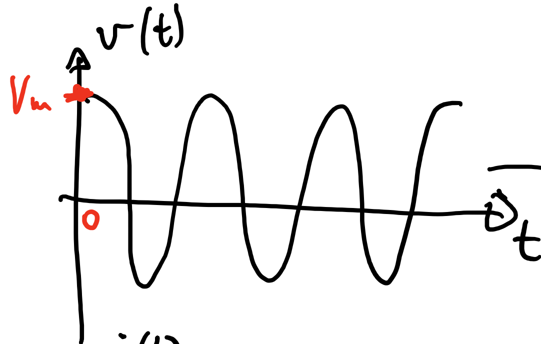


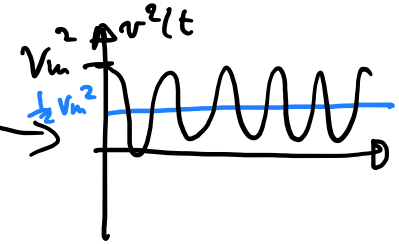
$$p = vi \qquad \langle p \rangle = \frac{1}{2} V_m I_m$$

$$v(t) = \boxed{V_m} \cos(\omega t) = \boxed{V_{rms}\sqrt{2}} \cos \omega t$$

$$i(t) = I_m \cos(\omega t) = I_{rms}\sqrt{2} \cos \omega t$$

$$V_m = I_m R$$

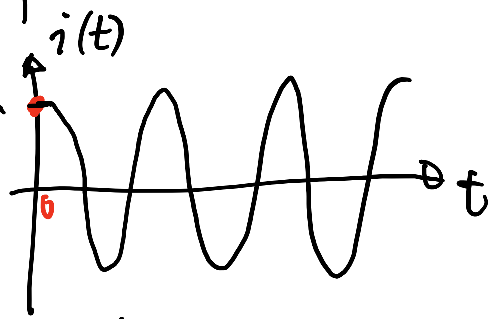




$$\rightarrow \sqrt{\tfrac{1}{2} V_m^2} \rightarrow V_{rms}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

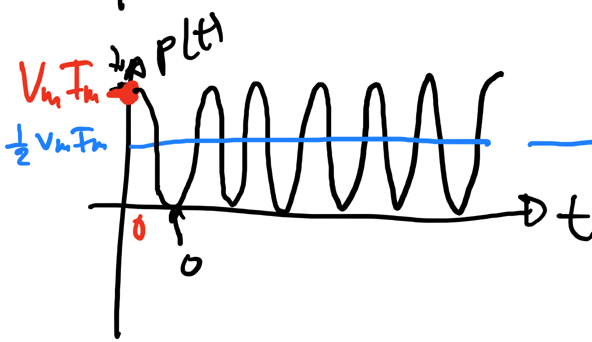
$$\frac{V_m}{R} = I_m$$



$$\langle p \rangle = \frac{1}{2} V_m I_m$$

$$= \cancel{\frac{1}{2}} V_{rms} \cancel{\sqrt{2}} I_{rms} \cancel{\sqrt{2}}$$

$$= V_{rms} I_{rms}$$



Power in phasor domain variables

$v(t) = 10\cos(\omega t - 30°)\ \text{V} \rightarrow \underline{V} = 10\angle{-30°} = 10e^{-j30°}$

$V_m = 10,\ \theta_v = -30°$

$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$

$Q = V_m \cos\theta_v$

$= Re\{V_m e^{j} \} = Re\{\underline{V}\}$

$P = Re\{\frac{1}{2} V_m I_m e^{j(\theta_v - \theta_i)}\}$

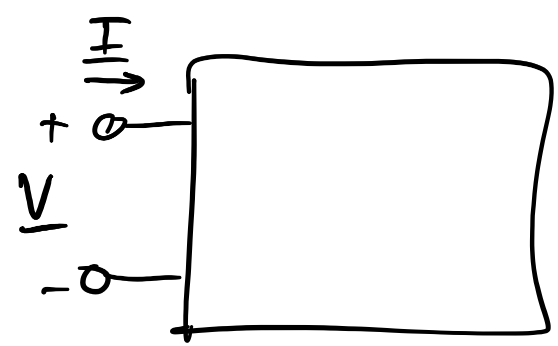
$= Re\{\frac{1}{2} V_m e^{j\theta_v} I_m e^{-j\theta_i}\}$

$$\boxed{P = Re\{\tfrac{1}{2}\, \underline{V}\ \underline{I}^*\}}$$

$$\boxed{\underline{V} = V_m e^{j\theta_v}}$$

$\underline{I} = I_m e^{j\theta_i}$

$\underline{I}^* = I_m e^{-j\theta_i}$

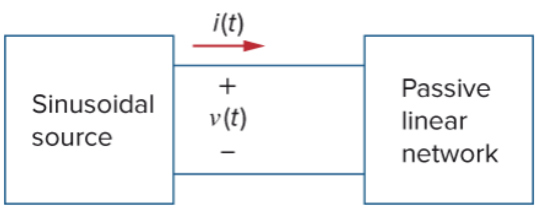


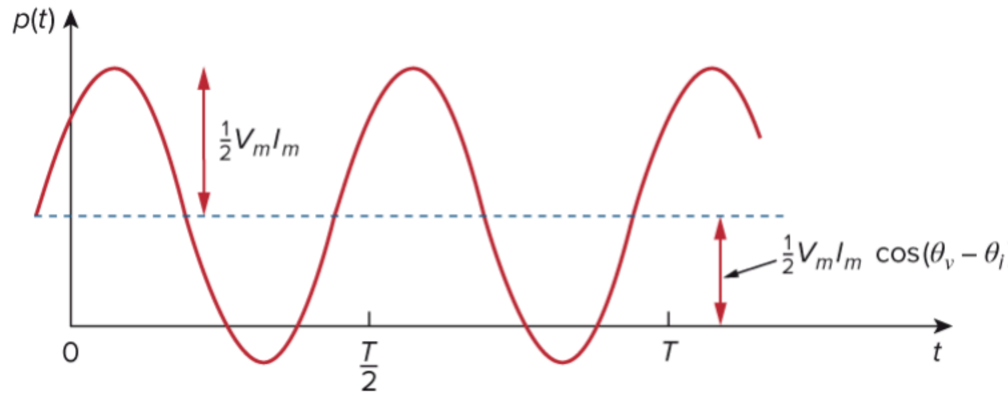
Power in phasor domain variables

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$





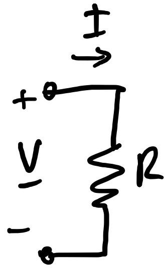
$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$\frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2} V_m I_m \underline{/\theta_v - \theta_i}$$

$$= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)]$$

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\mathbf{V}\mathbf{I}^*] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

$$P = \mathrm{Re}\left\{\frac{1}{2}\underline{V}\,\underline{I}^*\right\}$$

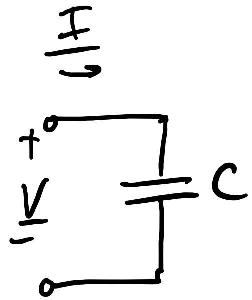


$$P = \mathrm{Re}\left\{\frac{1}{2}\underline{V}\,\underline{I}^*\right\}$$

$$P = \mathrm{Re}\left\{\frac{1}{2}\underline{I}R\,\underline{I}^*\right\}$$

$$= \frac{1}{2}R|\underline{I}|^2 = |\underline{I}_{rms}|^2 R$$

$$\boxed{\begin{aligned}\underline{X}\,\underline{X}^* &= x_m e^{j\theta} x_m e^{-j\theta} \\ &= |x_m|^2 \\ &= |\underline{X}|^2\end{aligned}}$$



$$P = \mathrm{Re}\left\{\frac{1}{2}\underline{V}\,\underline{I}^*\right\}$$

$$\underline{V} = Z\underline{I}$$

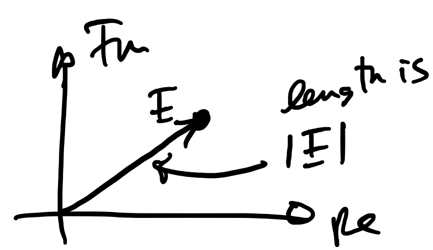
$$= \frac{1}{j\omega C}\underline{I}$$

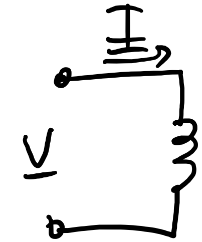
$$P = \mathrm{Re}\left\{\frac{1}{2}\frac{1}{j\omega C}\underline{I}\,\underline{I}^*\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{2}(-j)\frac{1}{\omega C}|\underline{I}|^2\right\}$$

$$= \mathrm{Re}\left\{0 + j\left(-\frac{1}{2}\frac{1}{\omega C}|\underline{I}|^2\right)\right\}$$

$$= 0$$





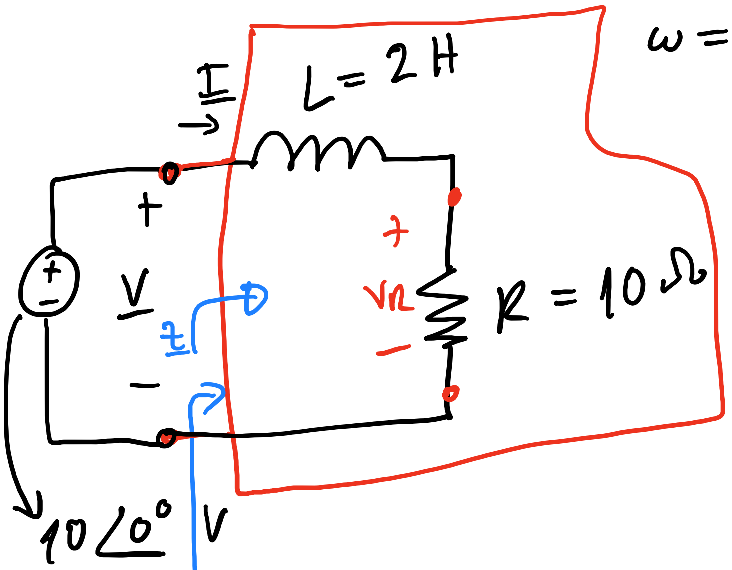
$$\underline{V} = Z\underline{I}$$

$$= j\omega L\,\underline{I}$$

$$P = \mathrm{Re}\left\{\frac{1}{2}j\omega L|\underline{I}|^2\right\}$$

$$= \mathrm{Re}\left\{\boxed{0} + j\boxed{\frac{\omega L}{2}|\underline{I}|^2}\right\}$$

$$= 0$$



$\omega = 5\,rad/s$

$= R + j\omega L = 10 + j5\cdot 2$

$Z_{in} = 10 + j10$

$P = \frac{1}{2} Re\{\underline{V}\,\underline{I}^*\} = \frac{1}{2} Re\{10\left(\frac{1+j1}{2}\right)\} = \frac{1}{2} Re\{5+j5\} = 2.5W$

$\underline{V} = 10\angle 0^\circ$

$\underline{I} = \frac{\underline{V}}{\underline{Z}} = \frac{10}{10+j10} = \frac{1}{1+j1}\,\frac{1-j1}{1-j1} = \frac{1-j1}{1^2-(j1)^2} = \frac{1-j1}{2}$

$a+b \quad a-b \quad a^2-b^2$

$P = \frac{1}{2} Re\left\{\underline{V}\left(\frac{\underline{V}}{\underline{Z}}\right)^*\right\}$

$= \frac{1}{2} Re\left\{\frac{|\underline{V}|^2}{\underline{Z}^*}\right\}$

$V_R = \frac{R}{R+j\omega L}\underline{V} = \frac{10}{10+j10}\underline{V} = \frac{1}{1+j1}\underline{V} = \frac{10}{1+j1} \rightarrow |\cdot|^2 = \frac{100}{2}$

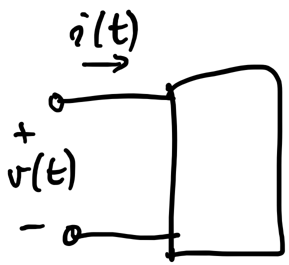
$P = \frac{1}{2} Re\left\{\frac{|\underline{V}|^2}{\underline{Z}^*}\right\} = \frac{1}{2} Re\left\{\frac{|V_R|^2}{R}\right\} = \frac{1}{2} Re\left\{\frac{100}{2\cdot 10}\right\} = 2.5W$

Practice Problem 11.1

Calculate the instantaneous power and average power absorbed by the passive linear network of **Fig. 11.1** if

$$v(t) = 330\cos(10t + 20^\circ)\text{ V} \qquad \text{and} \qquad i(t) = 33\sin(10t + 60^\circ)\text{ A}$$

Answer: $3.5 + 5.445\cos(20t - 10^\circ)$ kW, 3.5 kW.



$$p(t) = vi = 330\cos(10t+20^\circ)\cdot 33\cos(10t\underbrace{+60^\circ-90^\circ}_{-30^\circ})$$

$$330\angle 20^\circ \qquad\qquad 33\angle -30^\circ$$

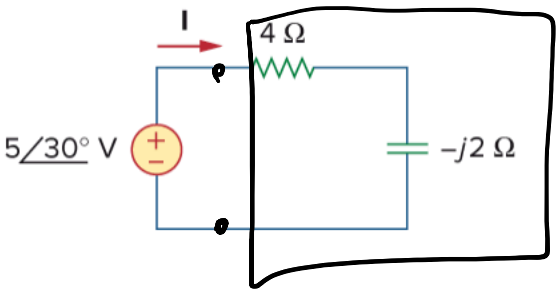
$$\underline{V}\,\underline{I}^* = (9900+990)\angle 50^\circ$$

$$\langle P\rangle = \tfrac{1}{2}\operatorname{Re}(VI^*) = \tfrac{1}{2}(10890\cos 50^\circ)$$

$$= 5495\cos 50^\circ$$

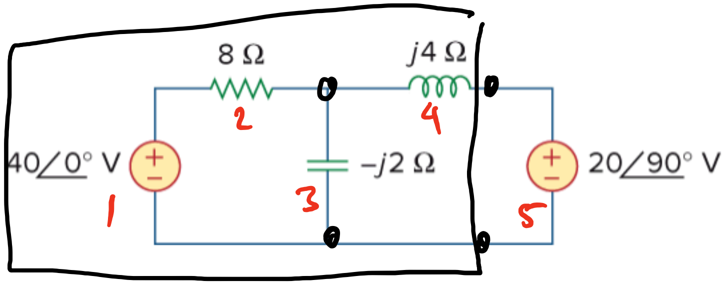
Example 11.3

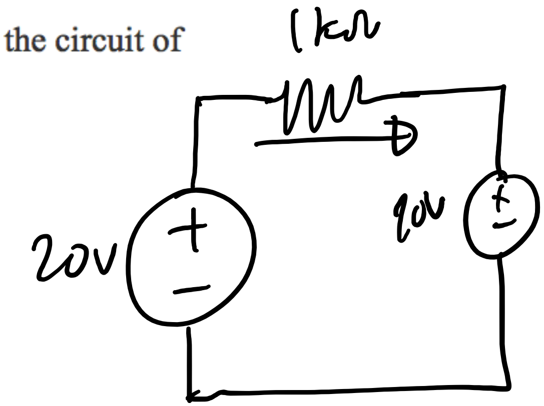
For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.



Practice Problem 11.4

Calculate the average power absorbed by each of the five elements in the circuit of **Fig. 11.6**.





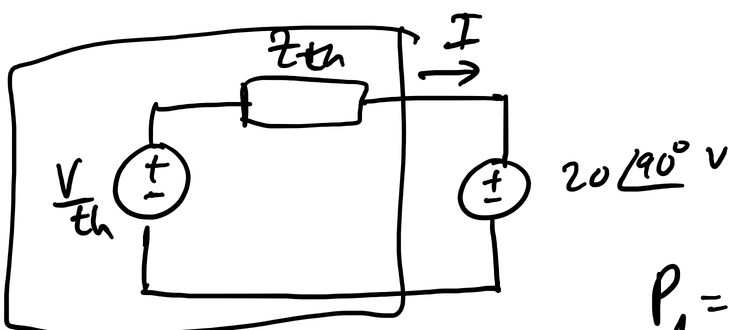
	+ve	0	−ve	unknown
1			✓	✓
2	✓			
3		✓		
4		✓		
5			✓	✓

$$V_{th} = \frac{-j2}{8-j2} 40\angle 0^\circ \ V$$

$$= 2.35 - j9.4 \ V$$

$$Z_{th} = 0.47 + j2.1 \ \Omega$$

$$I = \frac{V_{th} - 20\angle 90^\circ}{Z_{th}}$$

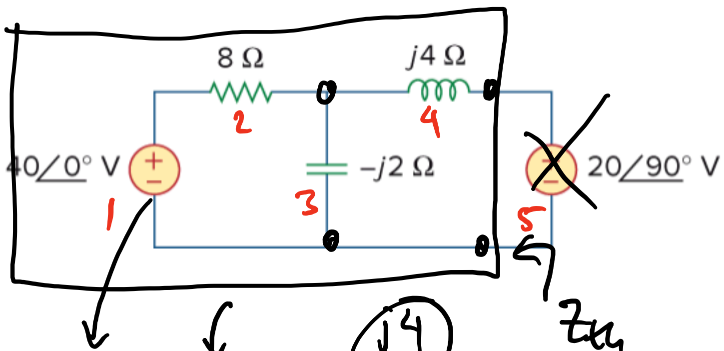


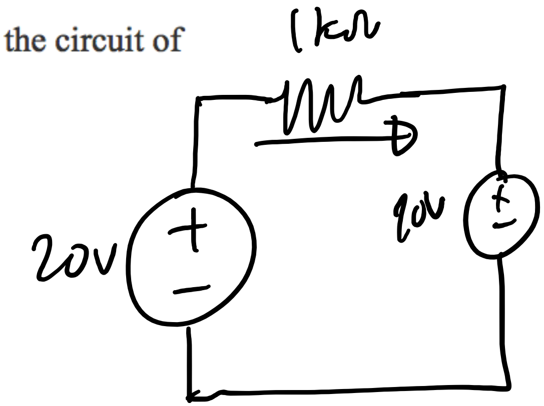
$$P_1 = \frac{1}{2} \mathrm{Re}\{V_{th}(I^*)\}$$

$$P_5 = \frac{1}{2} \mathrm{Re}\{20\angle 90^\circ I^*\}$$

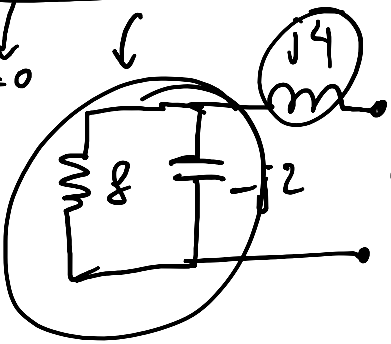
Practice Problem 11.4

Calculate the average power absorbed by each of the five elements in the circuit of **Fig. 11.6**.





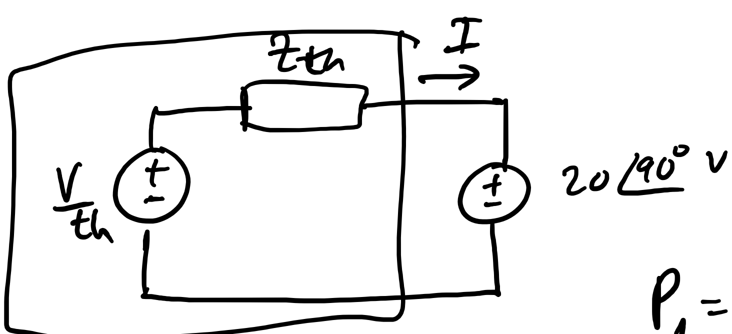
$V_s = 0$



$$Z_{th} = j4 + (8 \| -j2) = j4 + \left(\frac{1}{8} + \frac{1}{-j2}\right)^{-1}$$

$$\rightarrow Z_{th} = 0.47 + j2.1\ \Omega$$

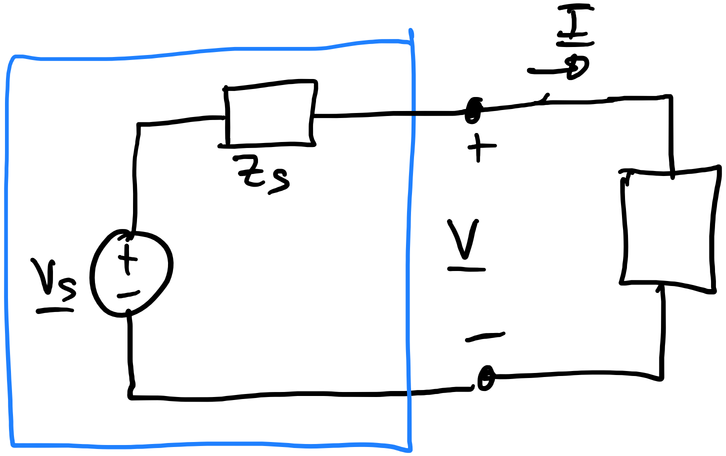
$$V_{th} = \frac{-j2}{8 - j2} 40\angle 0^\circ\ V = 2.35 - j9.4\ V$$



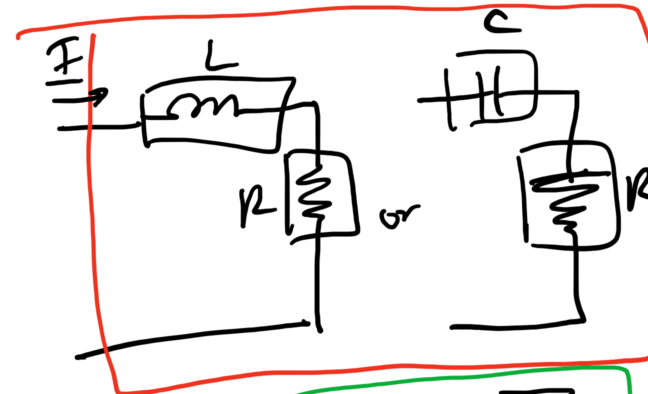
$$\underline{I} = \frac{\underline{V_{th}} - 20\angle 90^\circ}{Z_{th}}$$

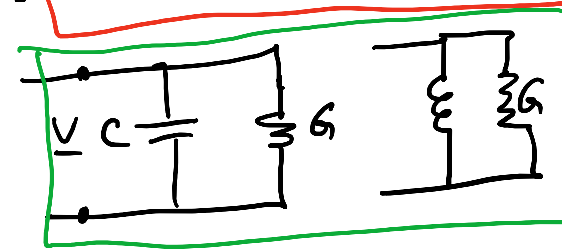
$$P_1 = \frac{1}{2} Re\{\underline{V_{th}}(\underline{I}^*)\}$$

$$P_5 = \frac{1}{2} Re\{20\angle 90^\circ I^*\}$$



$$Z_L = R + jX =$$





① $$\underline{V} = Z_L \underline{I}$$

② $$P = \frac{1}{2}\mathrm{Re}\{\underline{V}\,\underline{I}^*\} = \frac{1}{2}\mathrm{Re}\left\{\frac{|\underline{V}|^2}{Z_L^*}\right\} = \frac{1}{2}\mathrm{Re}\{Z_L|\underline{I}|^2\}$$

$$= \frac{1}{2}\mathrm{Re}\{(R+jX)|\underline{I}|^2\}$$

$$= \boxed{\frac{1}{2}R|\underline{I}|^2}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{|V|^2}{Z_L^*}\right\} = \frac{1}{2}\mathrm{Re}\{Y_L^*|V|^2\}$$

$$= \frac{1}{2}\mathrm{Re}\{(G+jB)^*|V|^2\}$$

$$= \boxed{\frac{1}{2}G|\underline{V}|^2}$$

$$\underline{V} = \frac{Z_L}{Z_s+Z_L}\underline{V_s}, \quad \underline{I} = \frac{\underline{V_s}}{Z_s+Z_L}$$

$$P = \frac{1}{2}\mathrm{Re}\left\{\frac{Z_L}{Z_s+Z_L}\underline{V_s}\,\frac{1}{(Z_s+Z_L)^*}\underline{V_s}^*\right\}$$

$$P = \frac{1}{2}\mathrm{Re}\left\{\frac{\overset{R+jX}{Z_L}}{\underset{\text{real}}{|Z_s+Z_L|^2}}\underset{\text{real}}{|\underline{V_s}|^2}\right\}$$

$$P = \frac{1}{2}\frac{R}{|Z_s+Z_L|^2}|\underline{V_s}|^2$$

$$P = \frac{1}{2}\,\frac{R_L}{\underbrace{|Z_s+Z_L|^2}}\,|V_s|^2$$

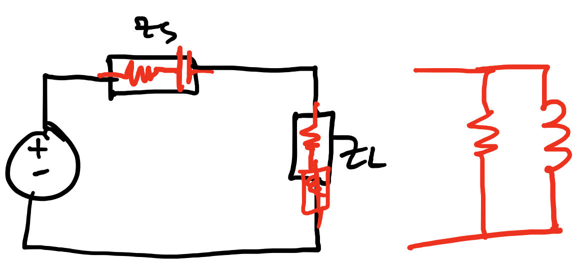
$$\frac{R_L}{|Z_s+Z_L|^2} = \frac{R_L}{(Z_s+Z_L)(Z_s+Z_L)^*} = \frac{R_L}{(R_s+jX_s+R_L+jX_L)(R_s-jX_s+R_L-jX_L)}$$

$$= \frac{R_L}{[R_s+R_L+j(X_s+X_L)][R_s+R_L-j(X_s+X_L)]} \to \frac{R_L}{(A+jB)(A-jB)}$$

$$= \frac{R_L}{A^2 \cancel{+jBA} \cancel{-jBA} + (jB)(-jB)} = \frac{R_L}{A^2+B^2}$$

$$= \frac{R_L}{(R_s+R_L)^2+(X_s+X_L)^2}$$

$$X_L = -X_s$$

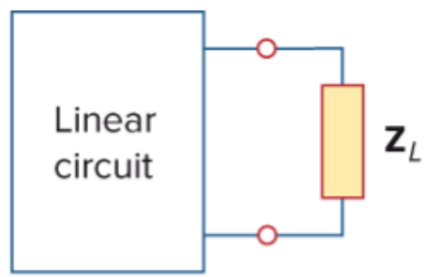


$$\Rightarrow \frac{R_L}{(R_s+R_L)^2} \text{ max when } R_L = R_s.$$

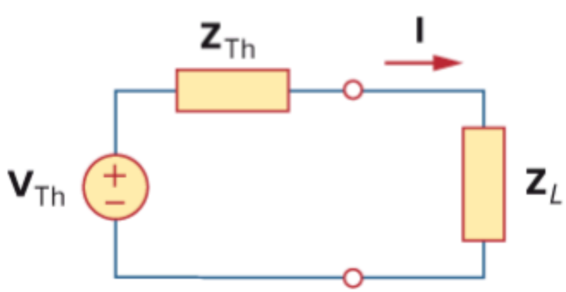
Maximum Average Power Transfer

$$\mathbf{Z}_{\mathrm{Th}} = R_{\mathrm{Th}} + jX_{\mathrm{Th}}$$

$$\mathbf{Z}_L = R_L + jX_L$$



(a)

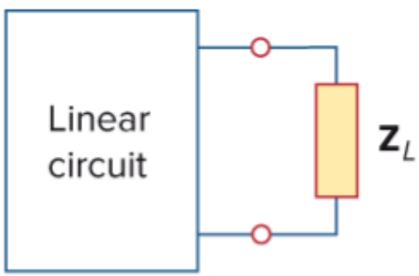


Maximum Average Power Transfer

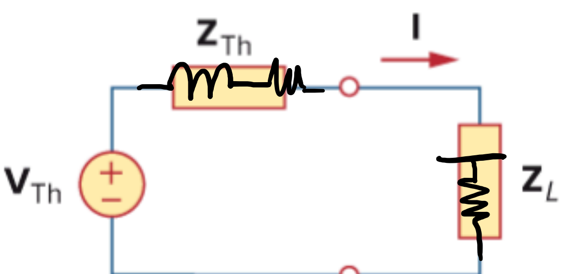
$$\mathbf{Z}_{Th} = R_{Th} + jX_{Th}$$

$$\mathbf{Z}_L = R_L + jX_L$$

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{Th}|^2 R_L/2}{(R_{Th}+R_L)^2 + (X_{Th}+X_L)^2}$$



(a)



$$P = \frac{1}{2}\frac{R_L}{|Z_s + Z_L|^2}|V_s|^2 = \frac{1}{2}\frac{R_s}{|Z_s + Z_s^*|^2}|V_s|^2$$

$$\boxed{Z_L = Z_s^*}$$

$$= \frac{1}{2}\frac{R_s}{|2R_s|^2}|V_s|^2$$

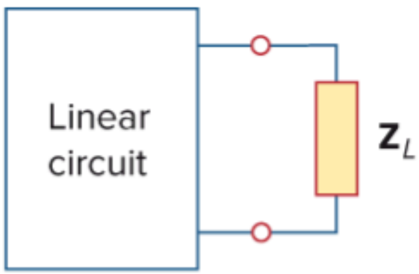
$$= \frac{1}{8}\frac{1}{R_s}|V_s|^2$$

$$\boxed{P_{max} = \frac{|V_s|^2}{8R_s}}$$

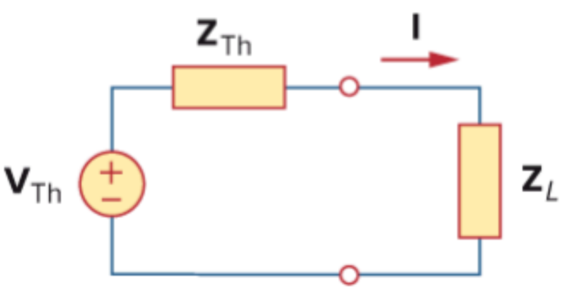
Maximum Average Power Transfer

$$\mathbf{Z}_{\mathrm{Th}} = R_{\mathrm{Th}} + jX_{\mathrm{Th}}$$

$$\mathbf{Z}_L = R_L + jX_L$$



(a)



$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{\mathrm{Th}}|^2 R_L/2}{(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2}$$

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\mathrm{Th}} - jX_{\mathrm{Th}} = \mathbf{Z}_{\mathrm{Th}}^*}$$

$$\boxed{P_{\max} = \frac{|\mathbf{V}_{\mathrm{Th}}|^2}{8R_{\mathrm{Th}}}}$$

Example 11.5

Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?