EK 307: Electric Circuits

Fall 2017

Lecture 20

Nov 21, 2017

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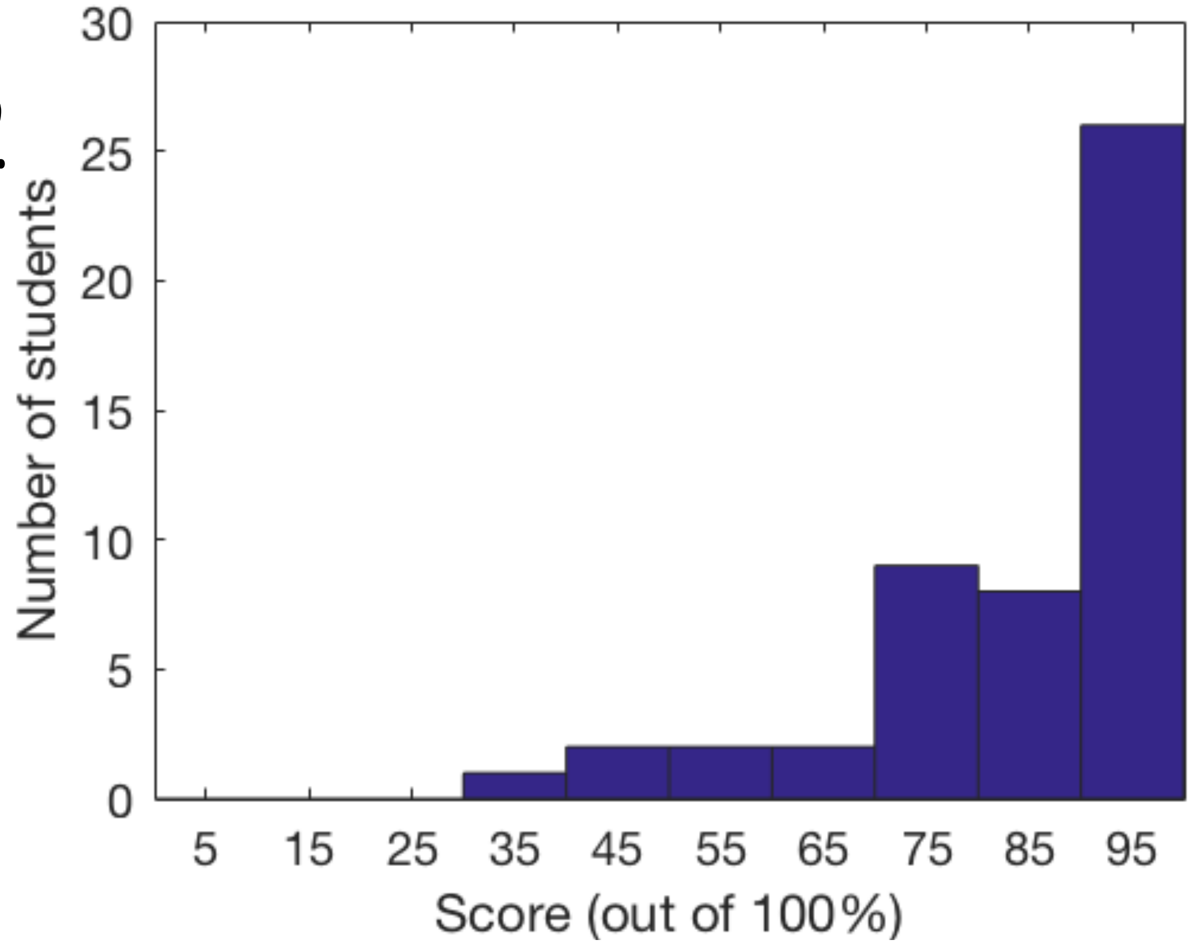
Administrivia

- Homework backlog cleared, hopefully all resolved
  - HW5,7,8 cleanup, back next Tues
- HW9 due next Tuesday

- Midterm #2: results

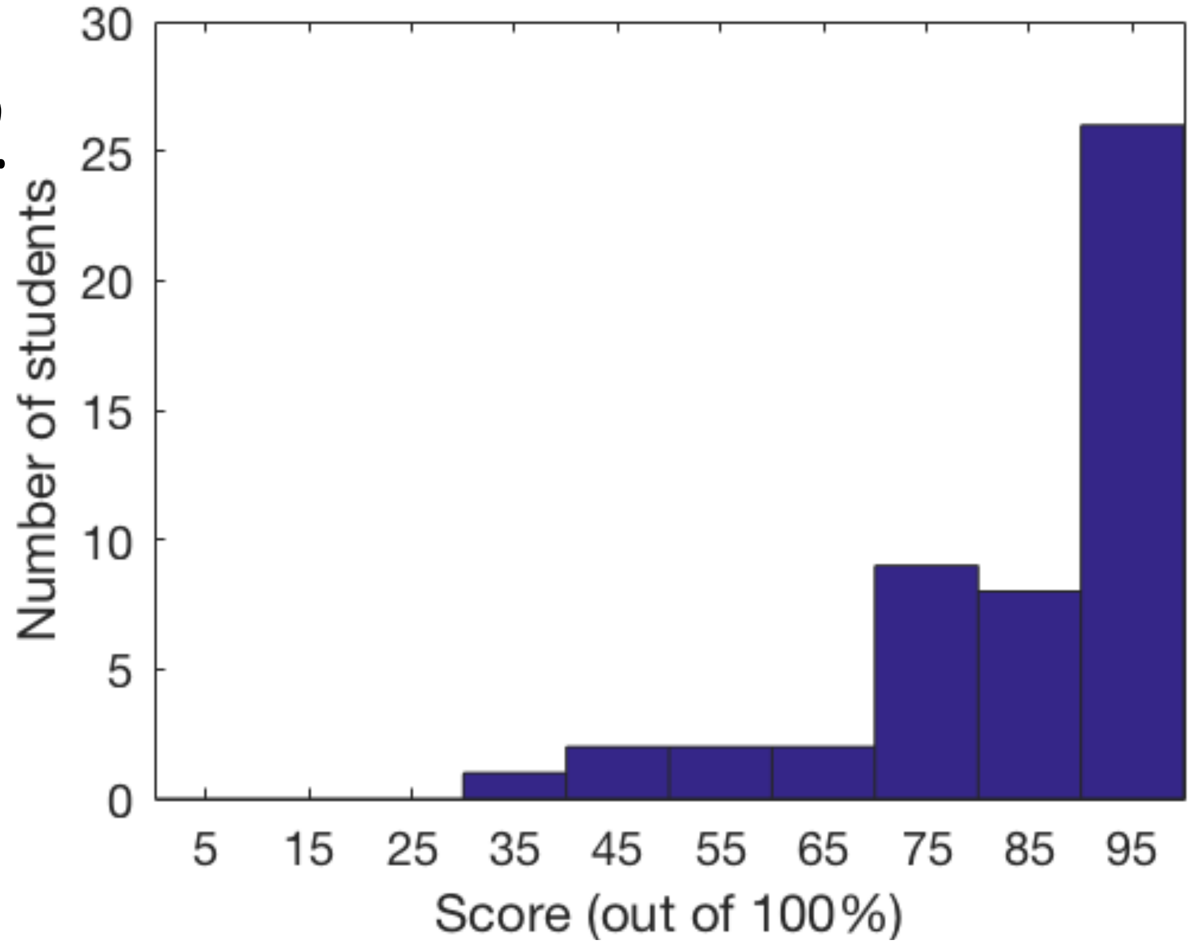
- Calibration: I will email out grade summary to date before Thanksgiving.

Midterm 2



	Midterm 2
Average:	**85%**
Standard deviation:	**15%**
Max score:	**100%**
Min score:	**38% (3 of 50 students below 50%)**

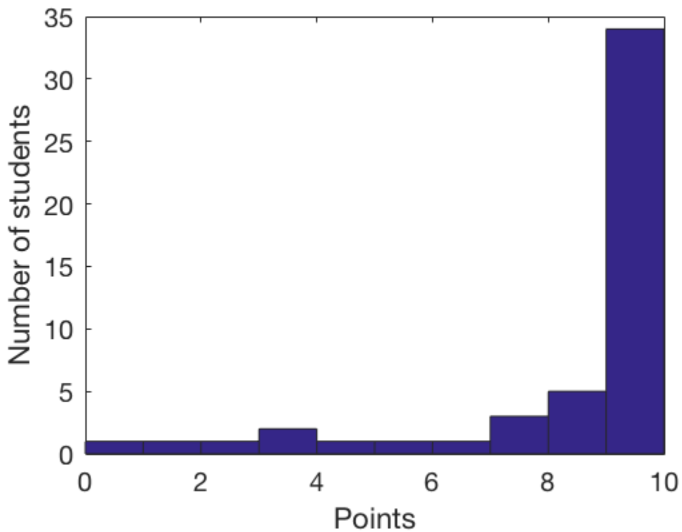
Midterm 2



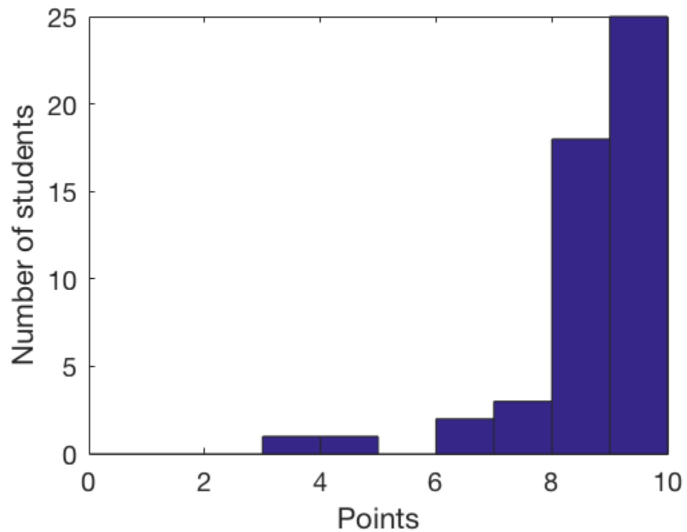
	Midterm 2	Midterm 1
Average:	**85% (B+)**	**72% (B-/C+)**
Standard deviation:	**15%**	**20%**
Max score:	**100%**	**98%**
Min score:	**38% (3 of 50 students below 50%)**	**26%**

Midterm 2

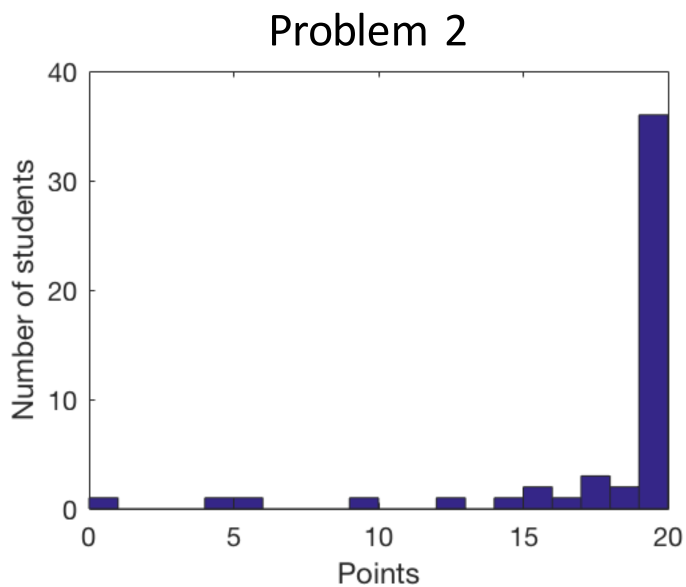
Multiple Choice

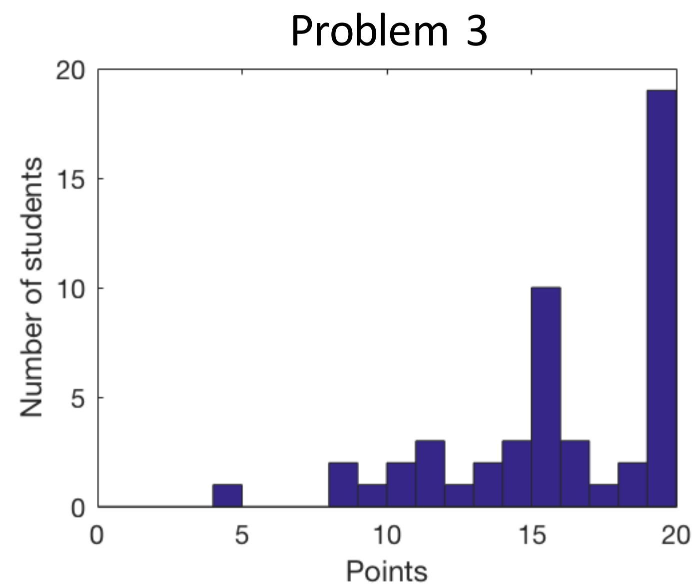


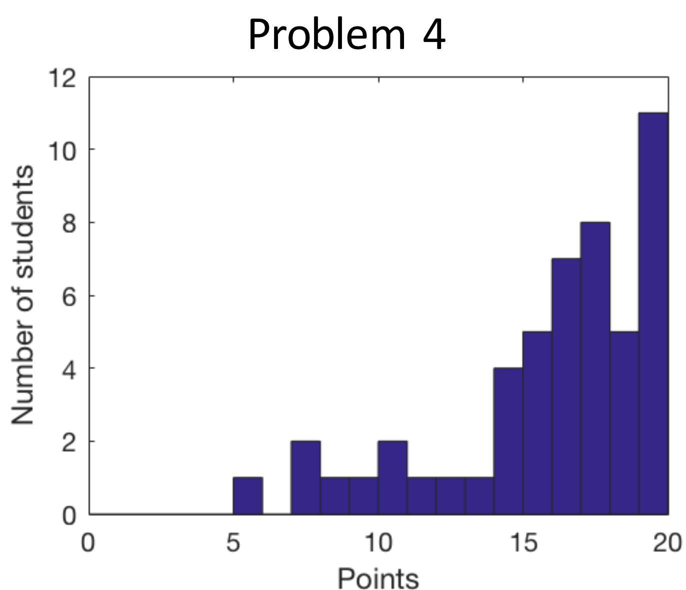
Problem 1

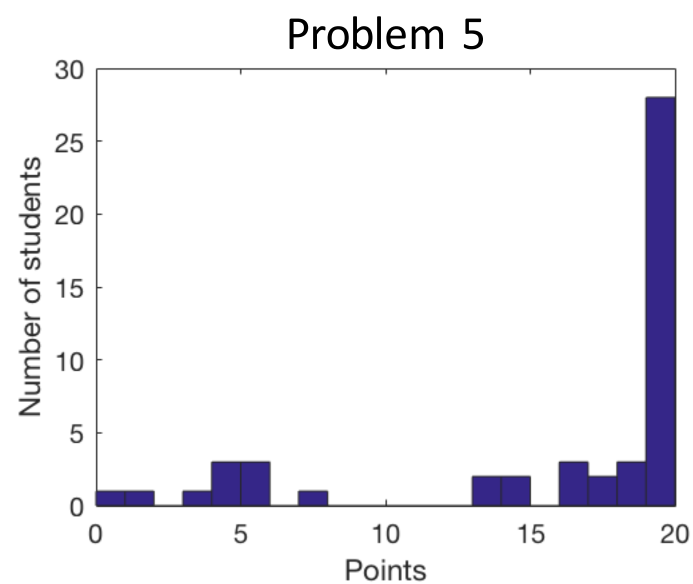


Midterm 2


Problem 2
40
30
20
10
0
Number of students
0
5
10
15
20
Points


Problem 3
20
15
10
5
0
Number of students
0
5
10
15
20
Points

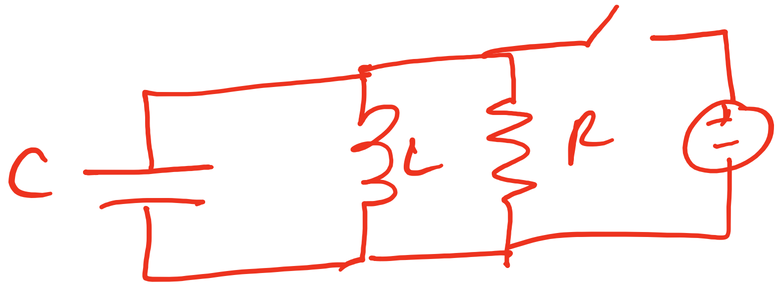

Problem 4
12
10
8
6
4
2
0
Number of students
0
5
10
15
20
Points


Problem 5
30
25
20
15
10
5
0
Number of students
0
5
10
15
20
Points

Last time (Lecture 19):

1. Second-order circuits & resonators

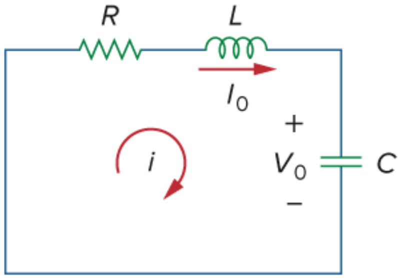
(Chapter 8)



Lecture 20: Chapter 8 – 2^{nd} order ccts

1. **Solving 2^{nd} order ccts – Sec 8.5-8.8**
2. **Chapter 9 – Phasors & Sinusoidal Steady State**

Summary



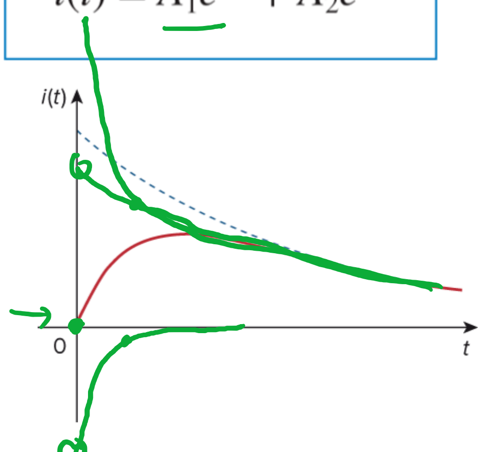
$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

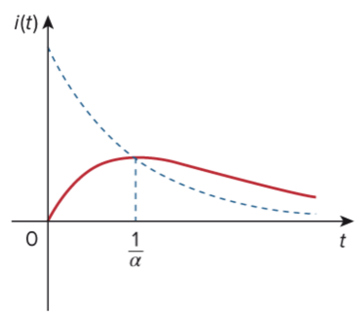
Overdamped

$$i(t) = \overset{-1}{\underline{A_1}} e^{s_1 t} + \overset{+1}{A_2} e^{s_2 t}$$



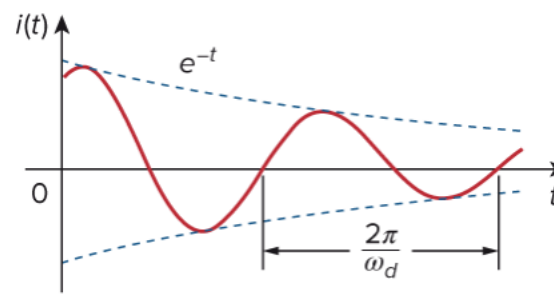
Critically damped

$$i(t) = (A_2 + A_1 t)e^{-\alpha t}$$



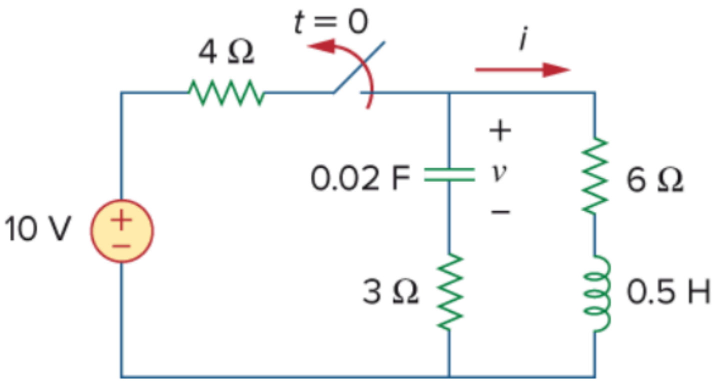
Underdamped

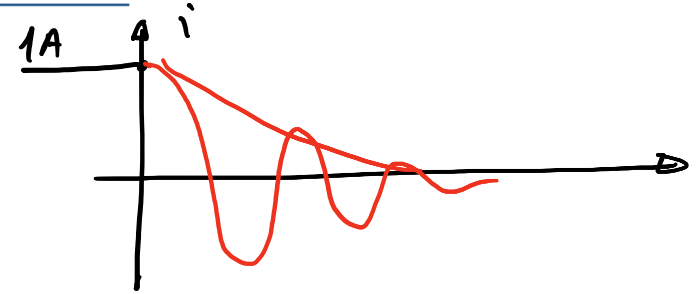
$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

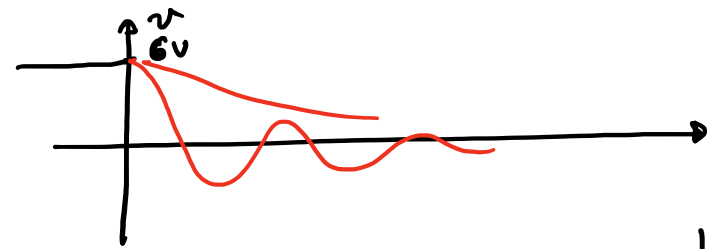


Example 8.4

Find $i(t)$ in the circuit of **Fig. 8.10**. Assume that the circuit has reached steady state at $t = 0^-$.







$2L$, $\omega_0 = \frac{1}{\sqrt{LC}}$

$= \frac{9\Omega}{2\times0.5H}$ $\quad = \frac{1}{\sqrt{0.01}}$

$= 9\ Np/s$ $\quad = 10\ rad/s$

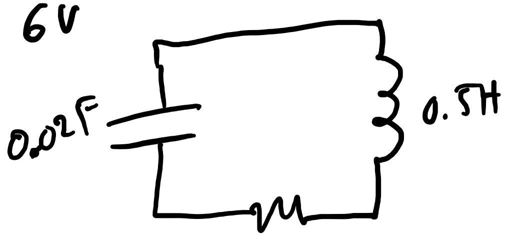
$\boxed{\alpha < \omega_0}$

underdamped

$i(0^+) = i = 1A$

$v(0^+) = v = 6V$

$t > 0$:



$v_L(t) = L\frac{di}{}$

$\cos\omega_0 t + B_2 \cancel{\sin}\,\omega_0 t)$

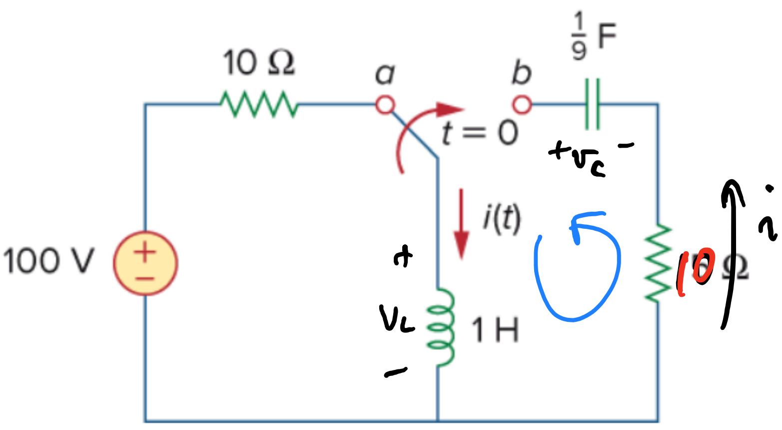
$B_1\omega_0 \cancel{\sin}\,\omega_0 t + B_2\omega_0\cos\omega_0 t$

$)]$

$= L(B_2\omega_0 - B_1\alpha) = 3V$

Practice Problem 8.4

The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.



$$i(0^-) = 10A$$

$$i(0^+) = 10A$$

$$v_c(0^-) = 0 = v_c(0^+)$$

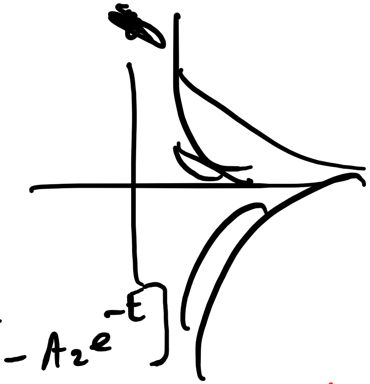
$$\alpha = \frac{R}{2L} = \frac{10}{2} = \cancel{2.5}\ 5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{9}}} = 3\ rad/s$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -\alpha \pm \sqrt{5^2 - 3^2}$$

$$i(t) = A_1 e^{\ } + A_2 e^{\ }$$



$$i(0) = 10A$$

$$= A_1 + A_2$$

$$v_L = L\frac{di}{dt} = L\left[(-9)A_1 e^{-9t} - A_2 e^{-t}\right]$$

At $t = 0^+$:

$$v_L(0^+) + i\,10 + (-v_c) = 0$$

$$v_L(0^+) = -i(0^+)\,10 + v_c(0^+)$$

$$= -100 + 0$$

→ At $t = 0^+ \Rightarrow -100$

$$1H(-9A_1 - (A_1 + A_2)) = -100$$

10

$$-8A_1 = -90$$

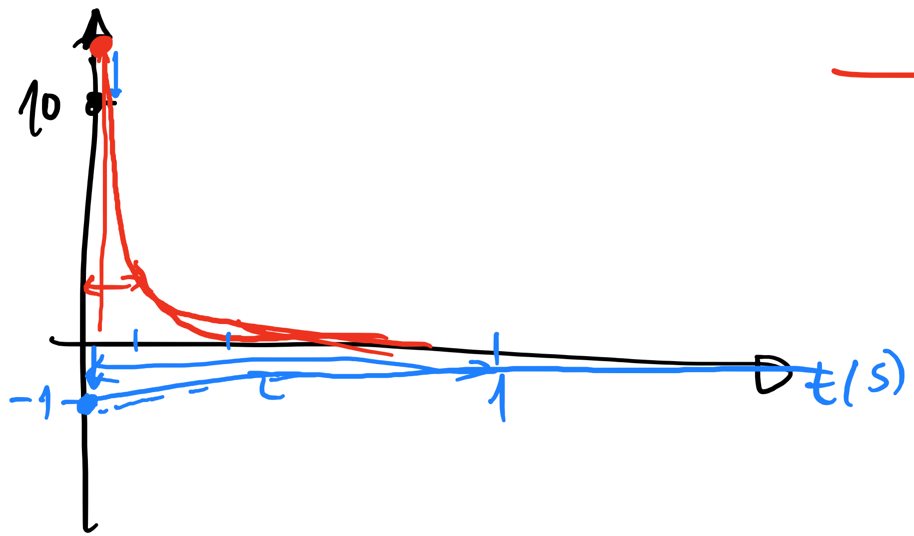
$$A_1 = \frac{90}{8} \sim 11$$

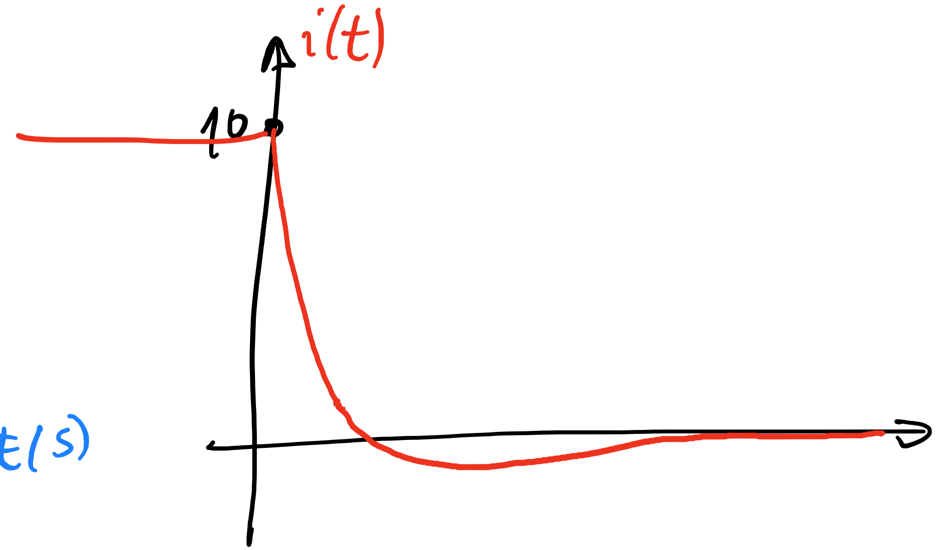
$$A_2 = 10 - 90/8 \sim -1$$

$$i(t) = 11\,e^{-9t} - 1\,e^{-t}$$

$$\tau_{1,2} = \frac{1}{9},\ 1 \qquad e^{-t/\tau}$$

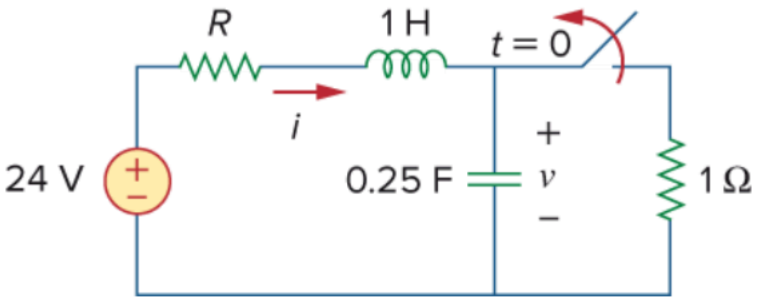
$$e^{-t/\frac{1}{9}\,s} \qquad e^{-t/1\,s}$$





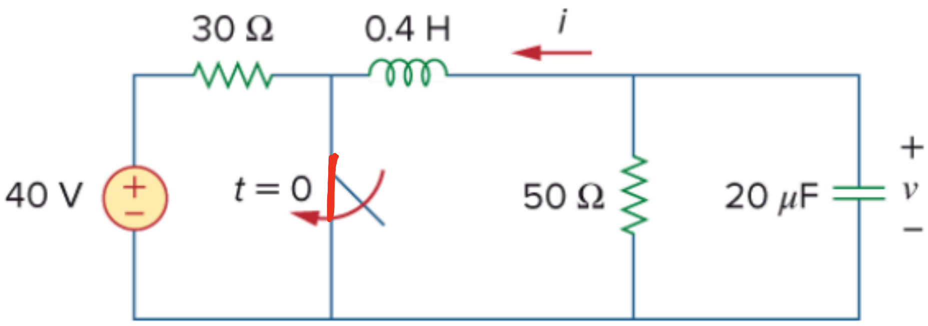
Example 8.7

For the circuit in **Fig. 8.19**, find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, and $R = 1\ \Omega$.



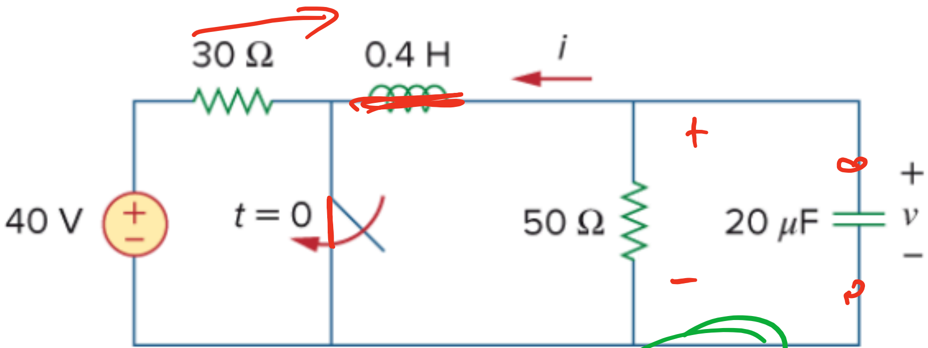
Example 8.6

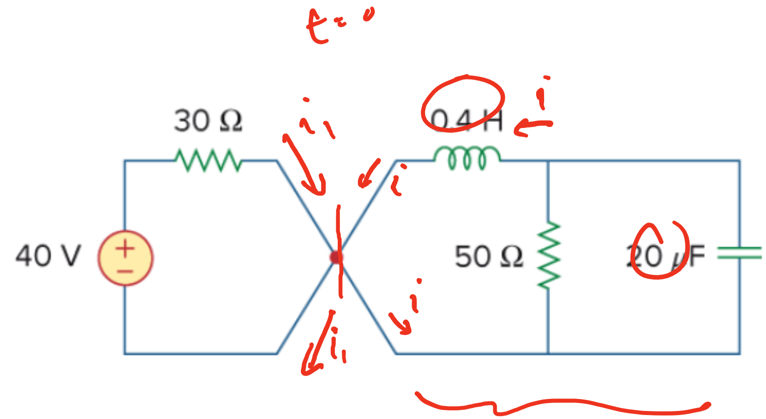
Find $v(t)$ for $t > 0$ in the *RLC* circuit of **Fig. 8.15**.



Example 8.6

Find $v(t)$ for $t > 0$ in the RLC circuit of Fig. 8.15.





At $t=0^-$, $v = \frac{50}{80}\cdot 40 = 25\,V$.

$i(0^-) = \boxed{i = -0.5A} = i(0^+)$

$\lambda = \frac{R}{2L}$, $\omega_0 = \frac{1}{\sqrt{LC}}$

$= \frac{50\,\Omega}{2\cdot 0.4H}$

$\lambda \lesseqgtr \omega_0$?

$i(t) = -0.5A,\ t<0$

For $t \geq 0$: $i(t) = e^{-\lambda t}(B_1 \cos\omega_0 t + B_2 \sin\omega_0 t)$

$$\left.\frac{di}{dt}\right|_{t=0} = -\lambda e^{-\lambda t}(B_1\cos\omega_0 t + B_2 \sin\omega_0 t) + e^{-\lambda t}(-B_1\omega_0 \sin\omega_0 t + B_2\omega_0\cos\omega_0 t)$$

$\boxed{\text{At } t=0:\ \frac{di}{dt} = -\lambda B_1 + B_2}$

$\boxed{i(0^+) = -0.5A = B_1}$

$\boxed{v = L\frac{di}{dt}} =$

$\boxed{v(0^+) = 25}$

Step response of RLC circuits

Step Response

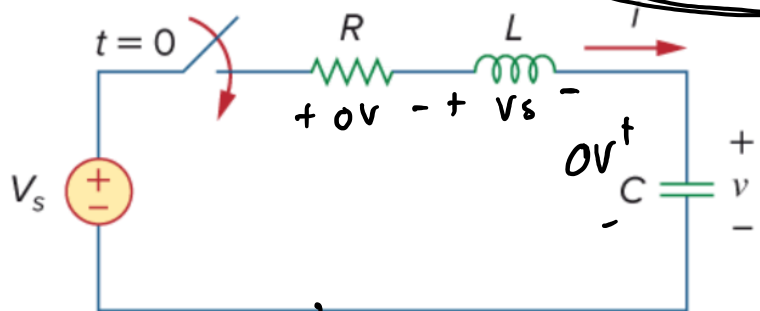
$v(0^-) = 0\ ,\ i(0^-) = 0$

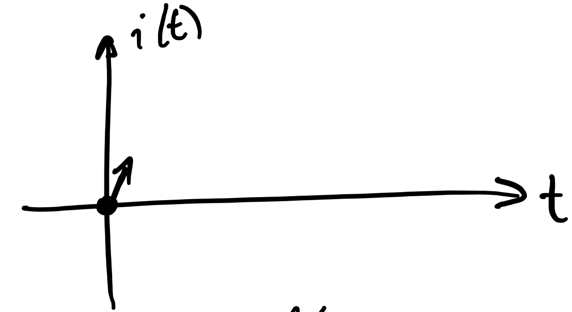
At $t = 0^+$: $\boxed{\underline{v(0^+)} = 0\ ,\ \underline{i(0^+)} = 0}$

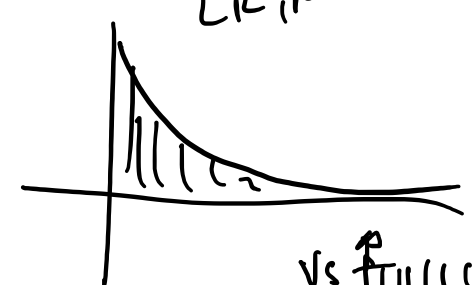
KVL $\rightarrow$ $L\frac{di}{dt} + Ri + v = V_s$

$\frac{di}{dt} = \frac{V_s}{L}$

$i = C\frac{dv}{dt}$





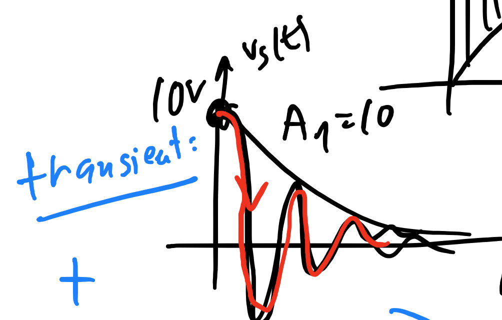


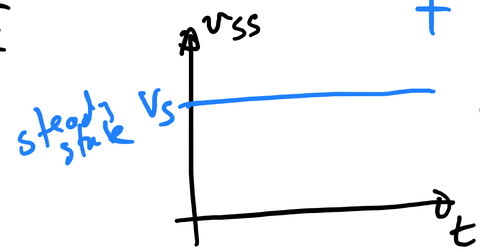
$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = V_s$

$$\boxed{\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}}$$

$s^2 + \frac{R}{L}s + \frac{1}{LC} = \frac{V_s}{LC}$

$$\boxed{V(\infty) = V_s}$$



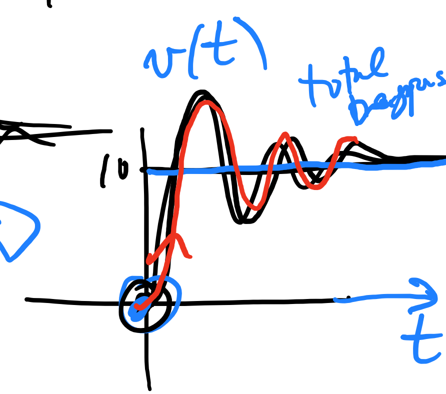


+

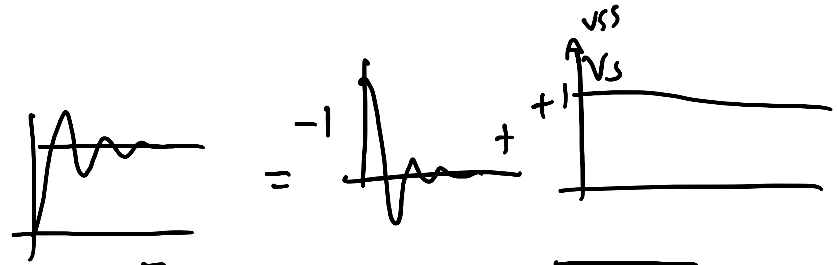
$V_s = 10$

$A_1 = -10$

$\Rightarrow$

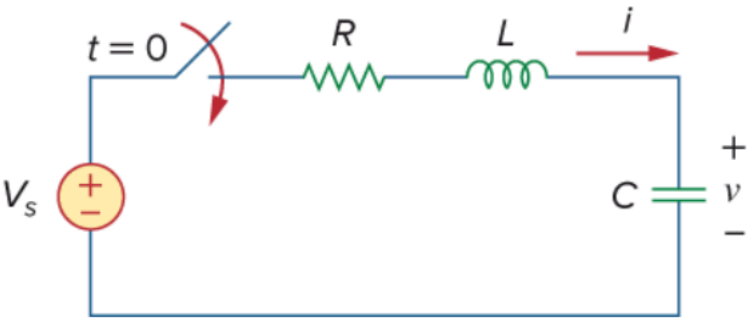


Step Response



$$v(t) = v_t(t) + \boxed{v_{ss}(t)}$$

$$L\frac{di}{dt} + Ri + v = V_s$$



$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)}$$

$$v_t(t) = (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$v_t(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$

$$\frac{d^2 v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

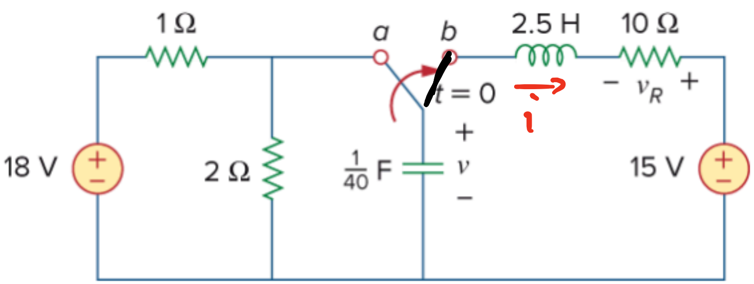
$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)}$$

$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$

Practice Problem 8.7

Having been in position *a* for a long time, the switch in **Fig. 8.21** is moved to position *b* at $t = 0$. Find $v(t)$ and $v_R(t)$ for $t > 0$.



$$v(0^-) = 12V = v(0^+)$$

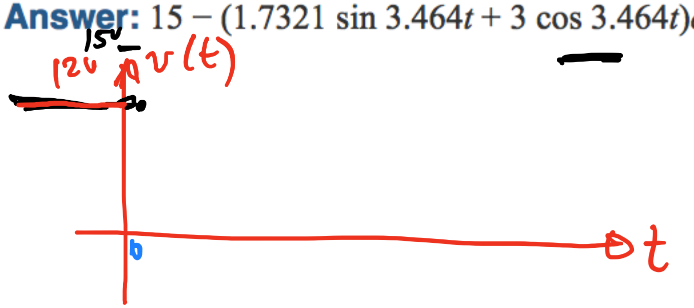
$$i(0^-) = 0 = i($$

$$v(\infty) = 15V$$

$$i(\infty) = 0$$

Figure 8.21 For **Practice Prob. 8.7.**

Answer: $15 - (1.7321 \sin 3.464t + 3 \cos 3.464t)e^{-2t}$ V, $3.464e^{-2t} \sin 3.464t$ V.



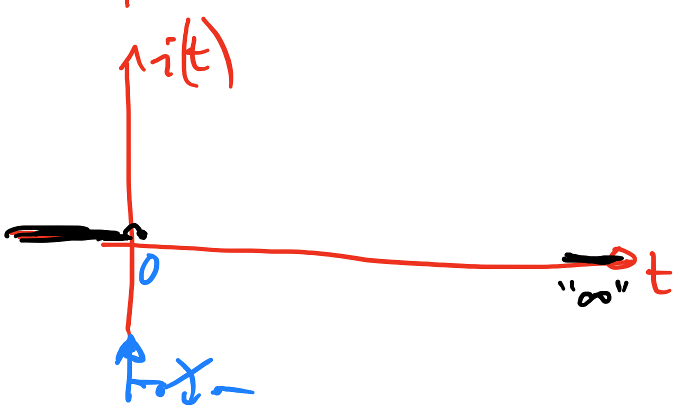
$$\boxed{\frac{d^2}{dt^2} i + \frac{R}{L}\frac{d}{dt} i + \frac{1}{LC} i = 0}$$

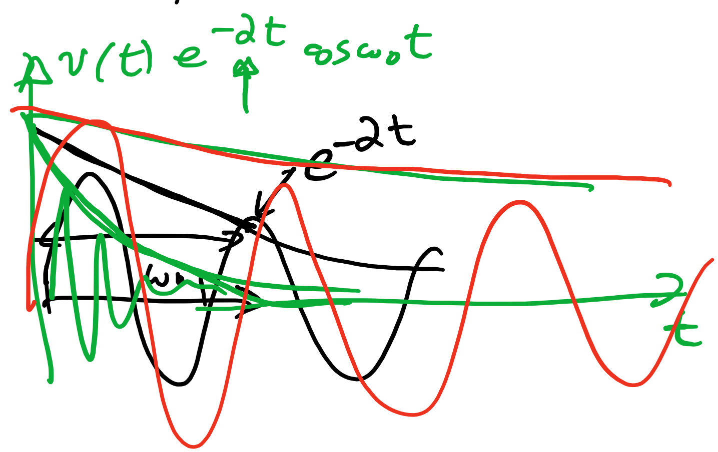
$$L\frac{di}{dt} + iR + \frac{1}{C}\int i\,dt' = 15$$

$$\alpha = \frac{R}{2L} = \frac{10}{2 \cdot 2.5} = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{40} \cdot 2.5}} = \sqrt{\frac{40}{2.5}}$$

$$\omega_0 = 4$$





Practice Problem 8.7

Having been in position a for a long time, the switch in **Fig. 8.21** is moved to position b at $t = 0$. Find $v(t)$ and $v_R(t)$ for $t > 0$.

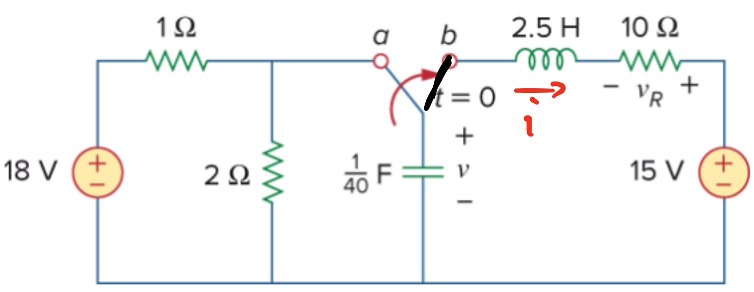


Figure 8.21 For **Practice Prob. 8.7**.

Answer: $15 - (1.7321 \sin 3.464t + 3 \cos 3.464t)e^{-2t}$ V, $3.464e^{-2t} \sin 3.464t$ V.

$\to v(0^-) = 12V = v(0^+)$

$i(0^-) = 0 = i($

$v(\infty) = 15V$

$i(\infty) = 0$

$$\boxed{\frac{d^2}{dt^2} i + \frac{R}{L}\frac{d}{dt} i + \frac{1}{LC} i = 0}$$

$L\frac{di}{dt} + iR +$

$\alpha = \frac{R}{2L} = \frac{10}{2 \cdot 2.5}$

$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{40} \cdot 2.5}} = \sqrt{\frac{40}{2.5}}$

$\omega_0 = 4$

$i(t) = e^{-\alpha t}(B$

$i(t) = e^{-\alpha t} B_2 \sin \omega_0 t$

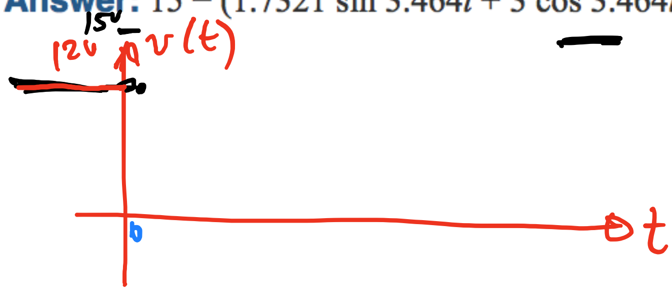
$= e^{-\alpha t} B_2 \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

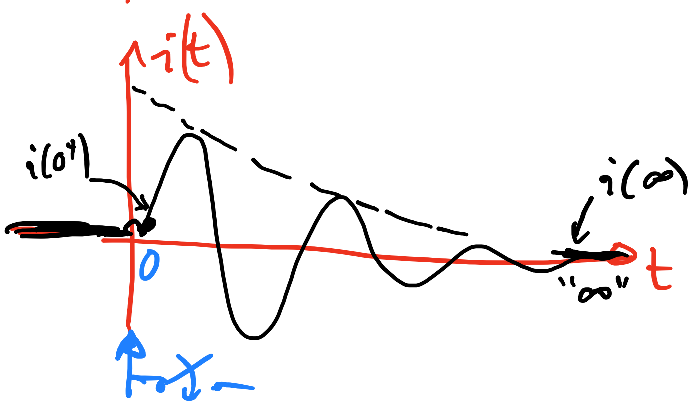
$= \frac{B_2}{2j} e^{(-\alpha + j\omega_0)t} - e^{(-\alpha - j\omega_0)t}$

$i = -C\frac{dv}{dt}$

$v = -\frac{1}{C}\int_0^t i\, dt' =$

$= -\frac{1}{C}\frac{B_2}{2j}\left[\frac{e^{(-\alpha + j\omega_0)t'}}{-\alpha + j\omega_0} - \frac{e^{(-\alpha - j\omega_0)t'}}{-\alpha - j\omega_0}\right]_0^t$





Practice Problem 8.7

Having been in position a for a long time, the switch in **Fig. 8.21** is moved to position b at $t = 0$. Find $v(t)$ and $v_R(t)$ for $t > 0$.

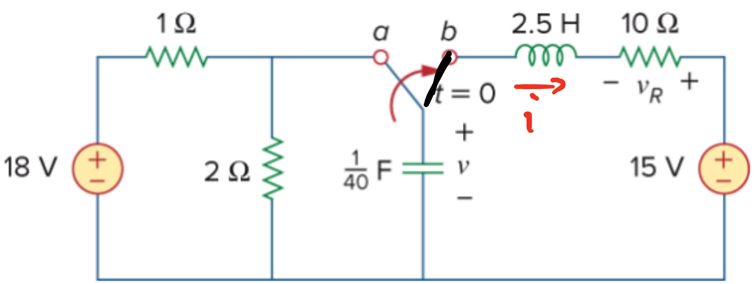
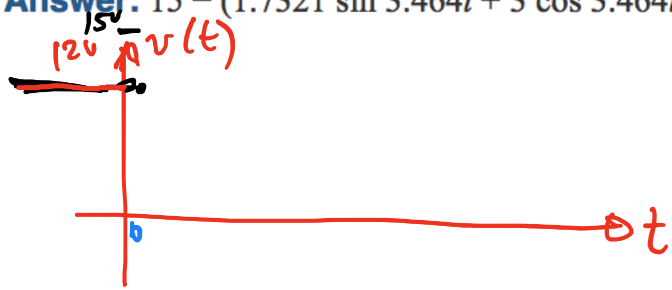
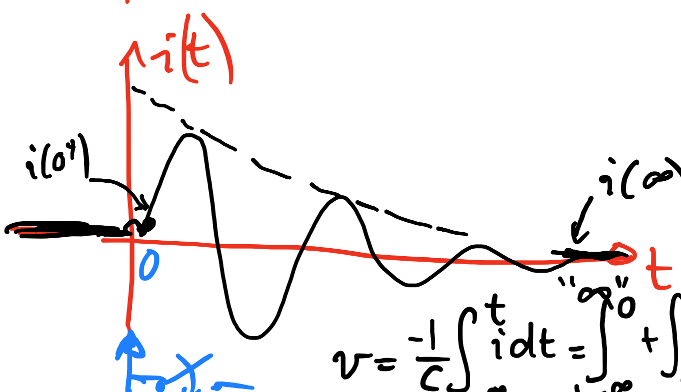


Figure 8.21 For **Practice Prob. 8.7.**

Answer: $15 - (1.7321 \sin 3.464t + 3 \cos 3.464t)e^{-2t}$ V, $3.464e^{-2t} \sin 3.464t$ V.





$(-2+j\omega_0)(-2-j\omega_0) = 2^2+\omega_0^2$

$v(0^+) = 12V = \frac{B_2}{C(2^2+\omega_0^2)}[\omega_0 - \omega_0] + v(0^+)$

$v(\infty) = \frac{B_2}{C(2^2+\omega_0^2)}(-\omega_0) + v(0^+) = 15V$

$v(t) = \frac{B_2 e^{-2t}}{C(2^2+\omega_0^2)}\left[\frac{e^{-2t}(2\sin\omega_0 t - 0 + \omega_0(\cos\omega_0 t - 1))}{2^2+\omega^2}\right]$

$= \frac{B_2}{C}\left[\frac{+2e^{-2t}\sin\omega_0 t' + \omega_0 e^{-2t}}{2^2+\omega_0^2}\right]_0^t$

$v = \frac{-B_2}{2jC}\left[\frac{-2(e^{(-2+j\omega_0)t'}(-2-j\omega_0) - \ldots)}{2^2+\omega_0^2}\right] - j\omega_0$

$v = \frac{-B_2}{2jC}\left[\frac{e^{(-2+j\omega_0)t'}(-2-j\omega_0)}{(-2+j\omega_0)} \quad \frac{e^{(-2-j\omega_0)t'}(-2+j\omega_0)}{}\right]$

$i = -C\frac{dv}{dt}$

$v = \frac{-1}{C}\int_{-\infty}^{t} i\,dt = \int_{-\infty}^{0} + \int_0^t i\,dt'$

$v = \frac{-1}{C}\int_0^t i\,dt' = \frac{-1}{C}\frac{B_2}{2j}\left[\frac{e^{(-2+j\omega_0)t'}}{-2+j\omega_0} - \frac{e^{(-2-j\omega_0)t'}}{-2-j\omega_0}\right]_0^t + v(0^+)$

Step Response of Parallel RLC

- Section 8.6
- Similar to series
- Please review on own

General 2nd order circuits

1. We first determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$, as discussed in **Section 8.2**.
2. We turn off the independent sources and find the form of the transient response $x_t(t)$ by applying KCL and KVL. Once a second-order differential equation is obtained, we determine its characteristic roots. Depending on whether the response is overdamped, critically damped, or underdamped, we obtain $x_t(t)$ with two unknown constants as we did in the previous sections.
3. We obtain the steady-state response as

$$x_{ss}(t) = x(\infty) \tag{8.51}$$

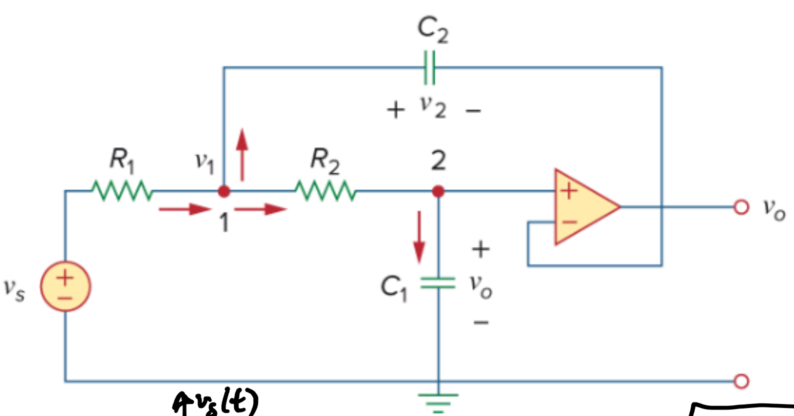
   where $x(\infty)$ is the final value of x, obtained in step 1.
4. The total response is now found as the sum of the transient response and steady-state response

$$x(t) = x_t(t) + x_{ss}(t) \tag{8.52}$$

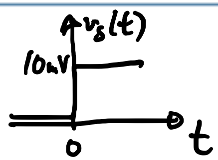
   We finally determine the constants associated with the transient response by imposing the initial conditions $x(0)$ and $dx(0)/dt$, determined in step 1.

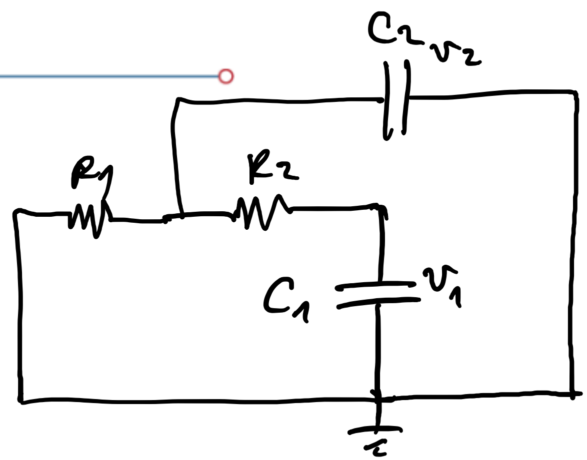
Example 8.11

In the op amp circuit of **Fig. 8.33**, find $v_o(t)$ for $t > 0$ when $v_s = 10u(t)$ mV. Let $R_1 = R_2 = 10\ \text{k}\Omega$, $C_1 = 20\ \mu\text{F}$, and $C_2 = 100\ \mu\text{F}$.



$v_s =$





At $t=0^-$:

$v_o = 0$

$v_2 = 0$

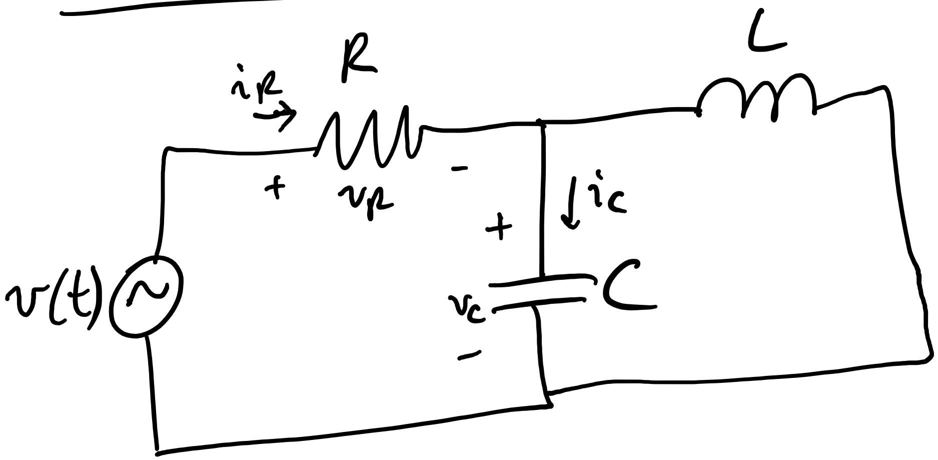
For $t>0$:

$v_o(t)$

Sinusoids and Phasors

Chapter 9

AC circuits



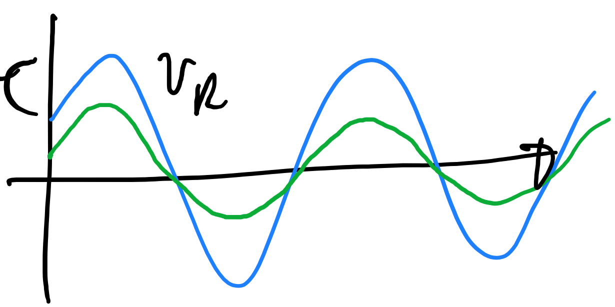
$$v_R = i_R R$$

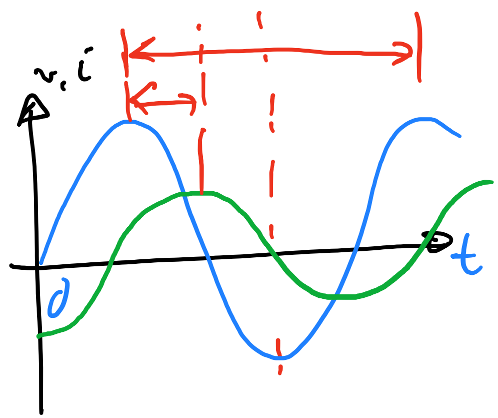
$$i_c = C\,dv$$

$$v_L = L\frac{di}{dt}$$

DC

AC





$$v(t) = Re\{\underline{e^{j\omega_0 t}}\} = Re\{\underline{\tilde{v}(t)}\} = \cos\omega_0 t$$

$$= Re\{\cos\omega_0 t + j\sin\omega_0 t\}$$

$$= \cos\omega_0 t$$

$$\tilde{i}(t) = C\frac{d\tilde{v}}{dt}$$

$$= C\frac{d}{dt}e^{j\omega_0 t}$$

$$= Cj\omega_0 e^{j\omega_0 t}$$

$$= Cj\omega_0 \tilde{v}$$

$$i(t) = C\frac{dv}{dt}$$

$$= C\frac{d}{dt}V_0\cos\omega_0 t$$

$$= -C\omega_0 V_0 \sin\omega_0 t$$

$$i(t) = Re\{\tilde{i}(t)\} = Re\{Cj\omega_0 e^{j\omega_0 t}\}$$

$$= C\omega_0 Re\{j(\cos\omega_0 t + j\sin\omega_0 t)\}$$

$$= -C\omega_0 \sin\omega_0 t$$

Sinusoids and Phasors

A **sinusoid** is a signal that has the form of the sine or cosine function.

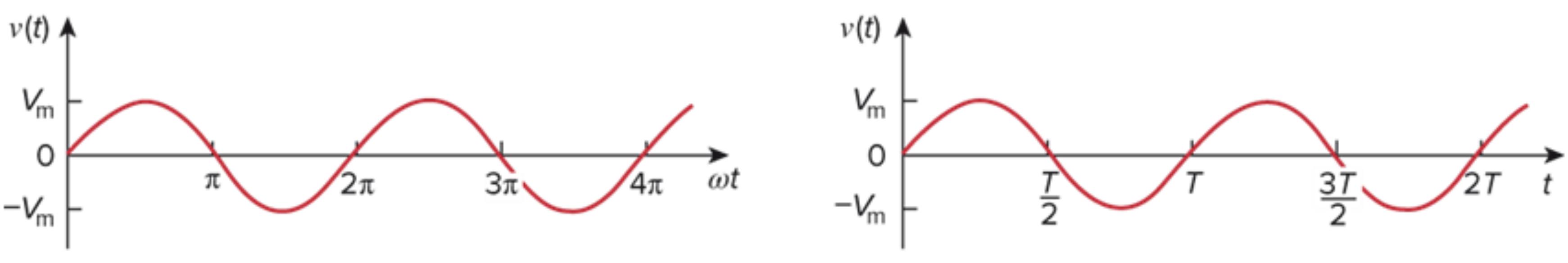
- aka AC

A **periodic function** is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

$$\boxed{T = \frac{2\pi}{\omega}}$$

$$v(t) = V_m \sin \omega t$$

rad/s



Sinusoids and Phasors

A **sinusoid** is a signal that has the form of the sine or cosine function.

A **periodic function** is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

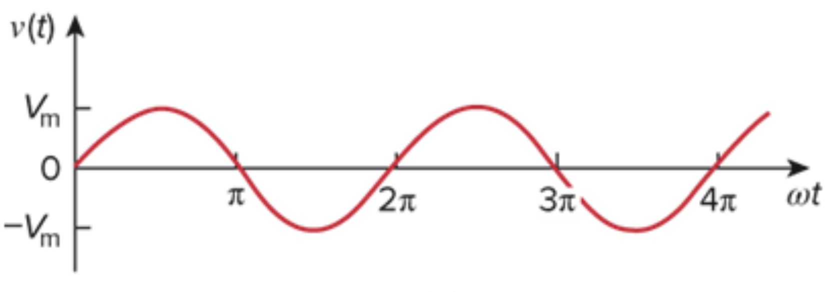
- aka AC

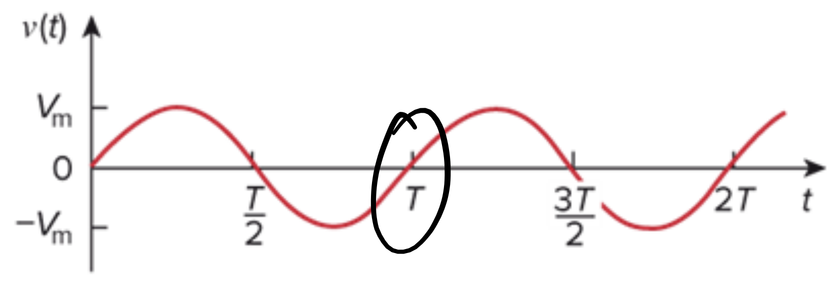
$$v(t) = V_m \sin \omega t$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

$$\boxed{f = \frac{1}{T}}$$

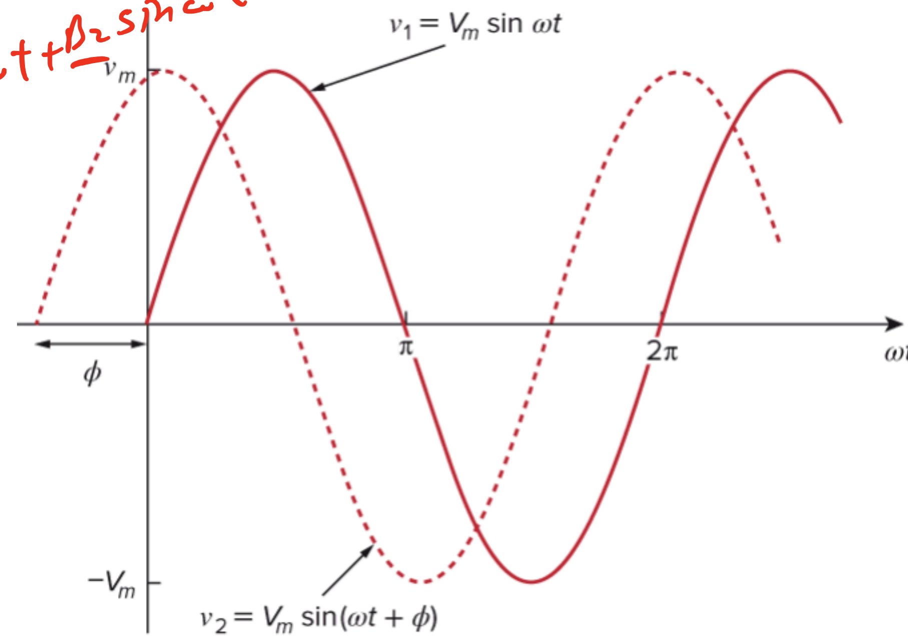
$$\omega = 2\,\pi f$$



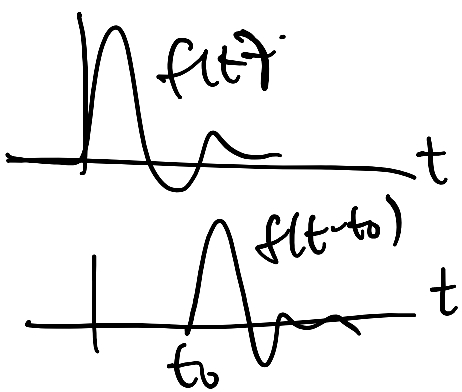


Sinusoids and Phasors

$$v(t) = V_m \sin(\omega t + \phi) = B_1 \cos\omega t + B_2 \sin\omega t$$



$$f(t - t_0)$$

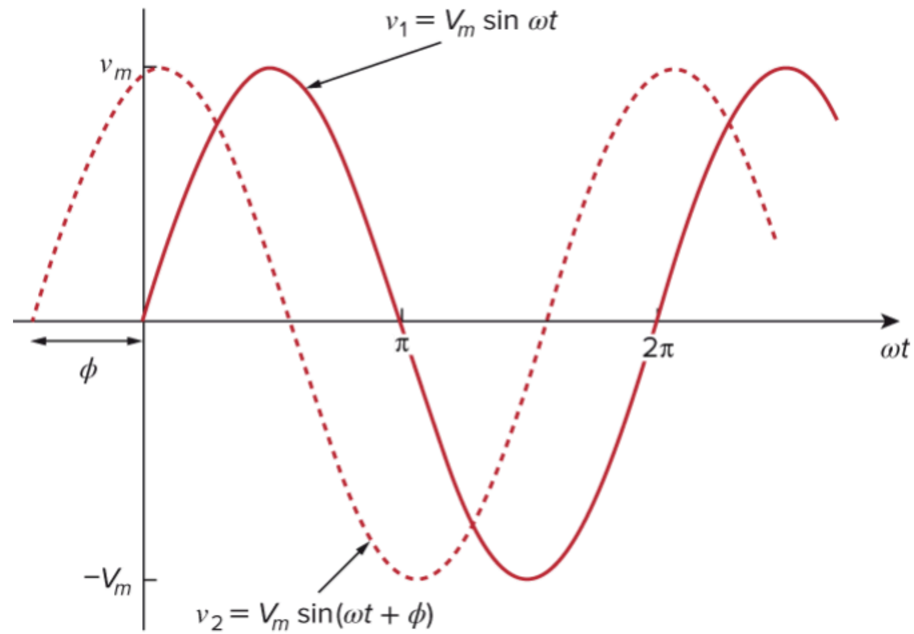


$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

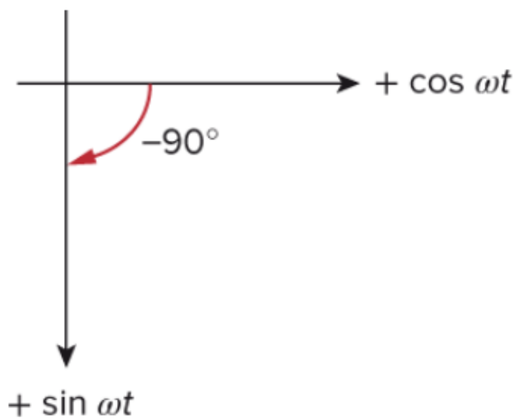
$$= \sin\omega t \underbrace{\cos\phi}_{B_1} + \cos\omega t \underbrace{\sin\phi}_{B_2}$$

Sinusoids and Phasors

$$v(t) = V_m \sin(\omega t + \phi)$$



$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$



Example 9.1

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12 \cos(50t + 10°) \text{ V}.$$

Phasors

A **phasor** is a complex number that represents the amplitude and phase of a sinusoid.

$$z = x + jy$$

$$z = r\underline{/\phi} = re^{j\phi}$$

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi}$$

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$
$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

$$\tilde{v}(t) = \underline{V}\, e^{j\omega_0 t}$$

$$= \overbrace{\left(V e^{j\theta}\right)}^{\underline{V}} e^{j\omega_0 t}$$

$$v(t) = \mathrm{Re}\{\tilde{v}(t)\} = \mathrm{Re}\{\underline{V}\, e^{j\omega_0 t}\}$$

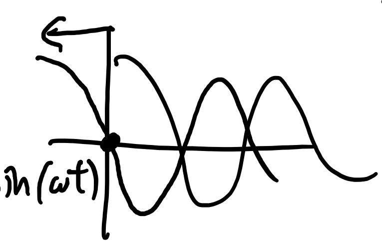
$$= \mathrm{Re}\{V e^{j\theta} e^{j\omega_0 t}\} = \mathrm{Re}\{V e^{j(\omega_0 t+\theta)}\}$$

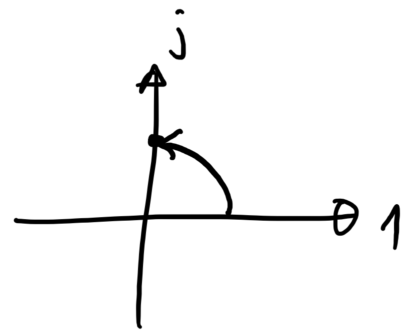
$$= V\,\mathrm{Re}\{\cos(\omega_0 t+\theta) + j\sin(\omega_0 t+\theta)\}$$

$$= V\cos(\omega_0 t+\theta)$$

$\underline{V}$	$v(t)$
1	(ωt)
$j = e^{j\frac{\pi}{2}}$	$\cos(\omega t+\frac{\pi}{2}) = -\sin(\omega t)$

90°





Phasors

$$v(t) = \mathrm{Re}(\mathbf{V}e^{j\omega t})$$

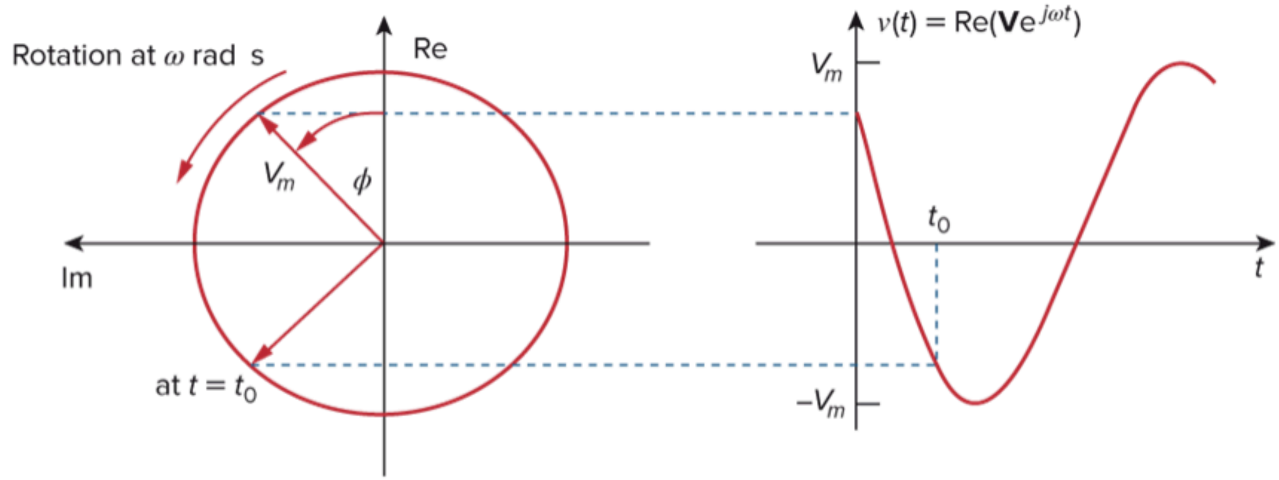
$$z = x + jy$$

$$z = r\angle\phi = re^{j\phi}$$

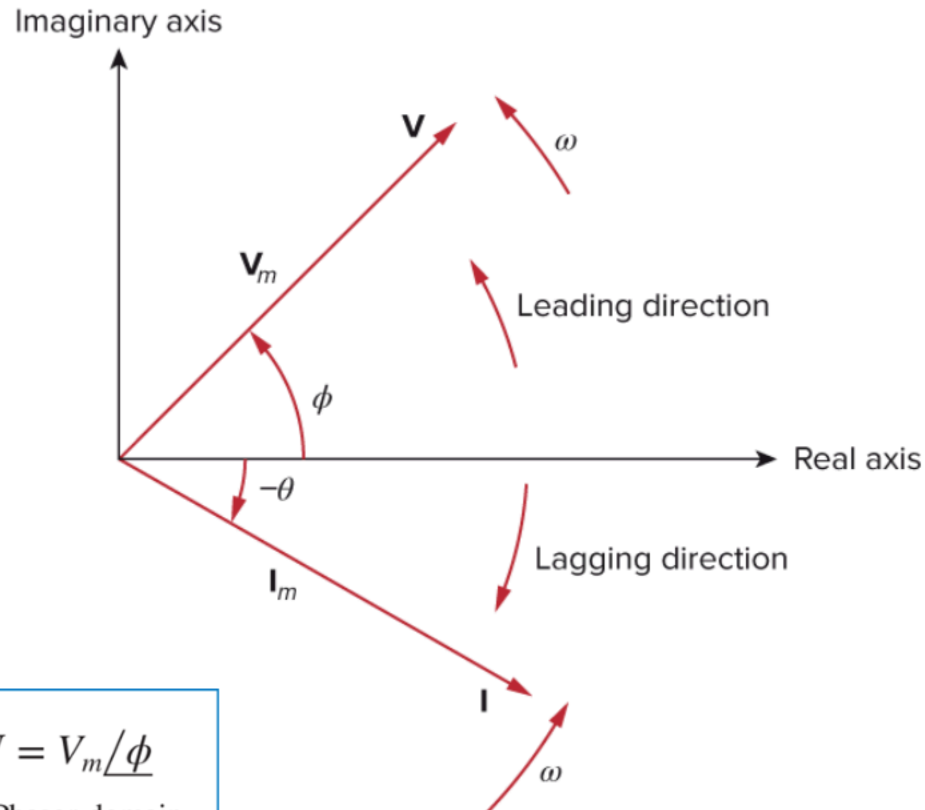
$$\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi$$

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$
$$\sin\phi = \mathrm{Im}(e^{j\phi})$$



Phase



$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

Phase

TABLE 9.1

Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

Circuit equations

$$\frac{dv}{dt} \quad \Leftrightarrow \quad j\omega\mathbf{V}$$

(Time domain) **(Phasor domain)**

$$\int v\,dt \quad \Leftrightarrow \quad \frac{\mathbf{V}}{j\omega}$$

(Time domain) **(Phasor domain)**

Sinusoids and Phasors