EK 307: Electric Circuits

Fall 2017

Lecture 18

Nov 9, 2017

Milos Popovic
Department of Electrical and Computer Engineering
Boston University

Administrivia

- Homework 7 due
- Midterm #2: Thu Nov 16

Last time (Lecture 17):

1. **Driven RC and RL circuits**
2. **First-order op-amp circuits**

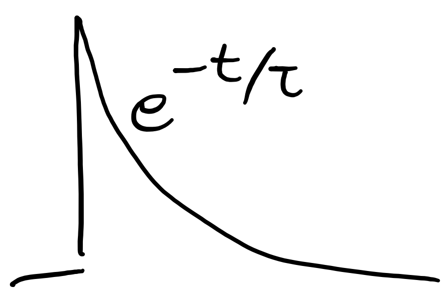
$i = C\frac{dv}{dt}$

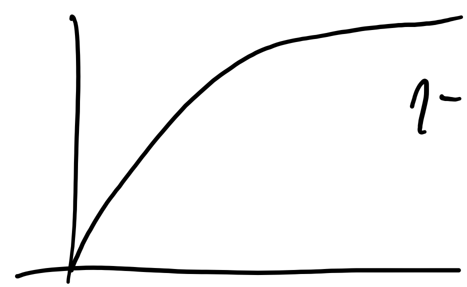
① initial $v_c(0^-)$ or $i_L(0^-)$

② SS $v_c(\infty)$, $i_L(\infty)$

③ RC.

(Chapter 7)





$1 - e^{-t/\tau}$

RC

$\frac{L}{R}$

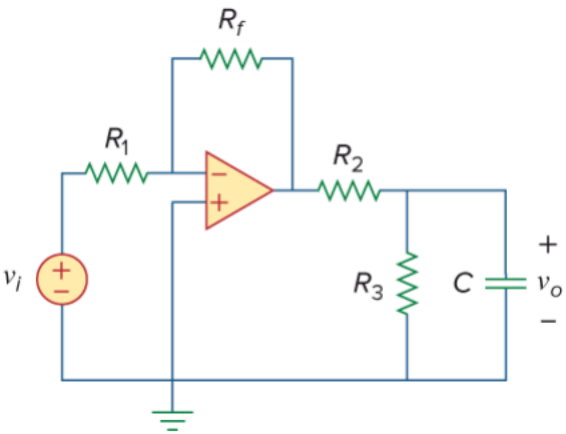
Lecture 18: Chapter 8 – 2^{nd} order ccts

1. **Driven RC and RL circuits**
2. **First-order op-amp circuits**

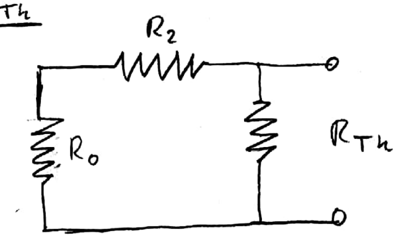
(covering sections 7.5-7.7)

Example 7.16

Find the step response $v_o(t)$ for $t > 0$ in the op amp circuit of **Fig. 7.59**. Let $v_i = 2u(t)$ V, $R_1 = 20$ kΩ, $R_f = 50$ kΩ, $R_2 = R_3 = 10$ kΩ, $C = 2$ μF.

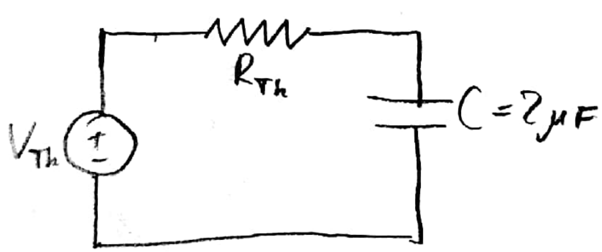


R_{Th}



$$R_{Th} = \frac{R_2 R_3}{R_2 + R_3}$$

equivalent circuit



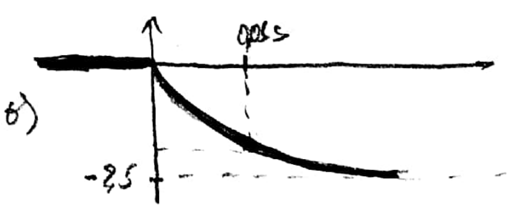
$$V_{Th} = -\frac{10}{20} \cdot \frac{50}{20} \cdot 2u(t) = -2{,}5u(t)$$

$$R_{Th} = \frac{R_2 R_3}{R_2 + R_3} = \frac{10 \cdot 10}{20} k\Omega = 5 k\Omega$$

$$\tau = R_{Th} \cdot C = 5 \cdot 10^3 \Omega \cdot 2 \cdot 10^{-6} F = 0{,}01 s$$

$$v_o(t) = v_o(\infty) + [v_o(0) - v_o(\infty)] e^{-\frac{t}{0{,}01s}}, \quad t > 0$$

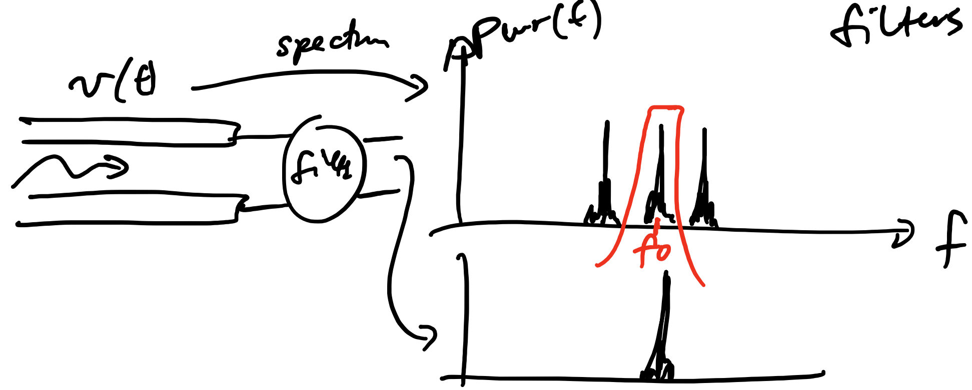
$$v_o(t) = -2{,}5 + [0 + 2{,}5] e^{-\frac{t}{0{,}01s}} = -2{,}5\left(1 - e^{-\frac{t}{0{,}01s}}\right) u(t)$$

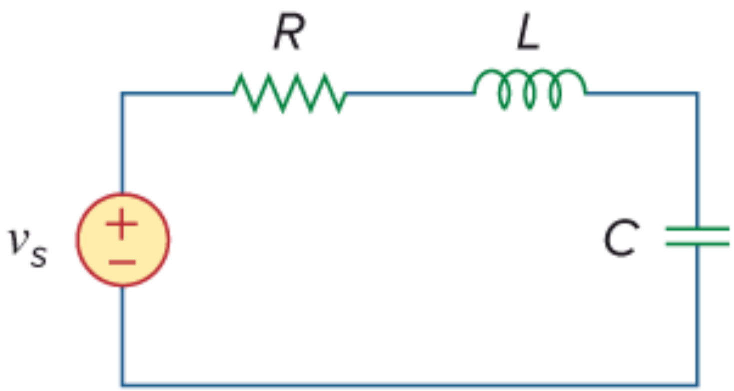


Second-order circuits

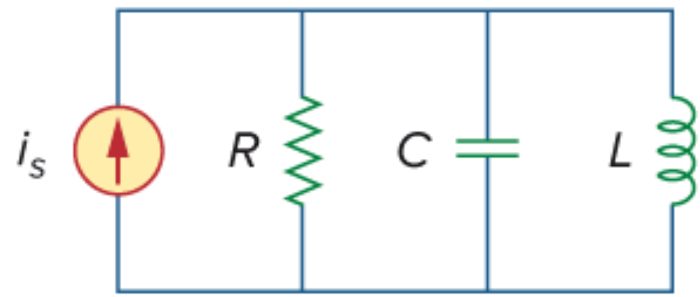
- 1^{st}-order circuits => 1^{st} order diff eq. => 1 energy storage element
- 2^{nd}-order circuits => 2^{nd} order diff eq. => 2 energy storage elements

Examples



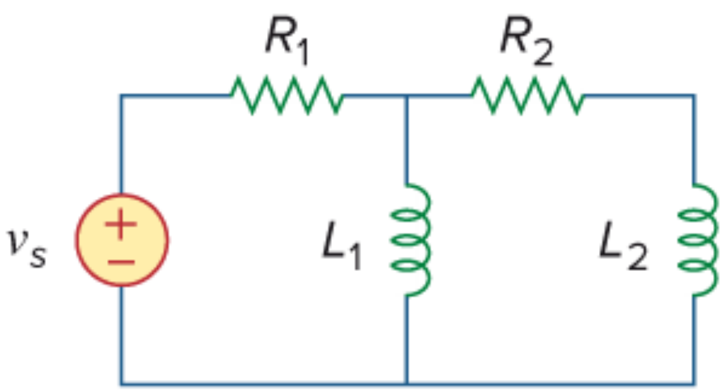


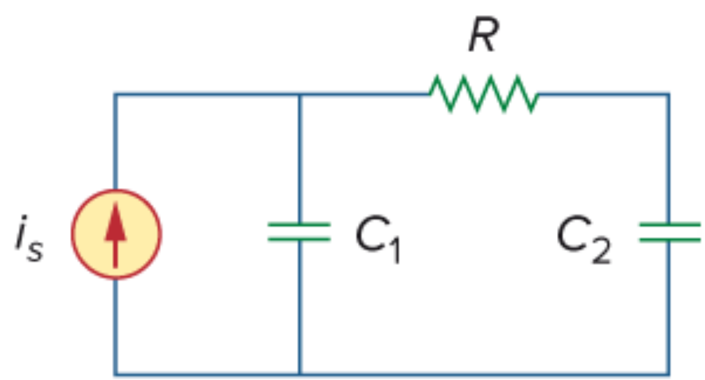
series RLC



parallel RLC

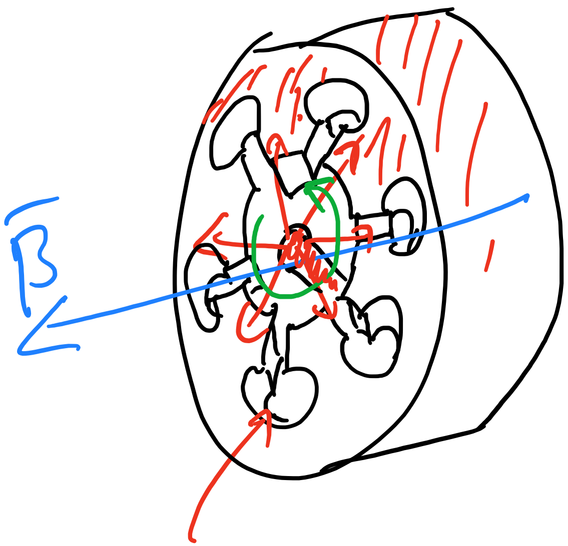
resonators



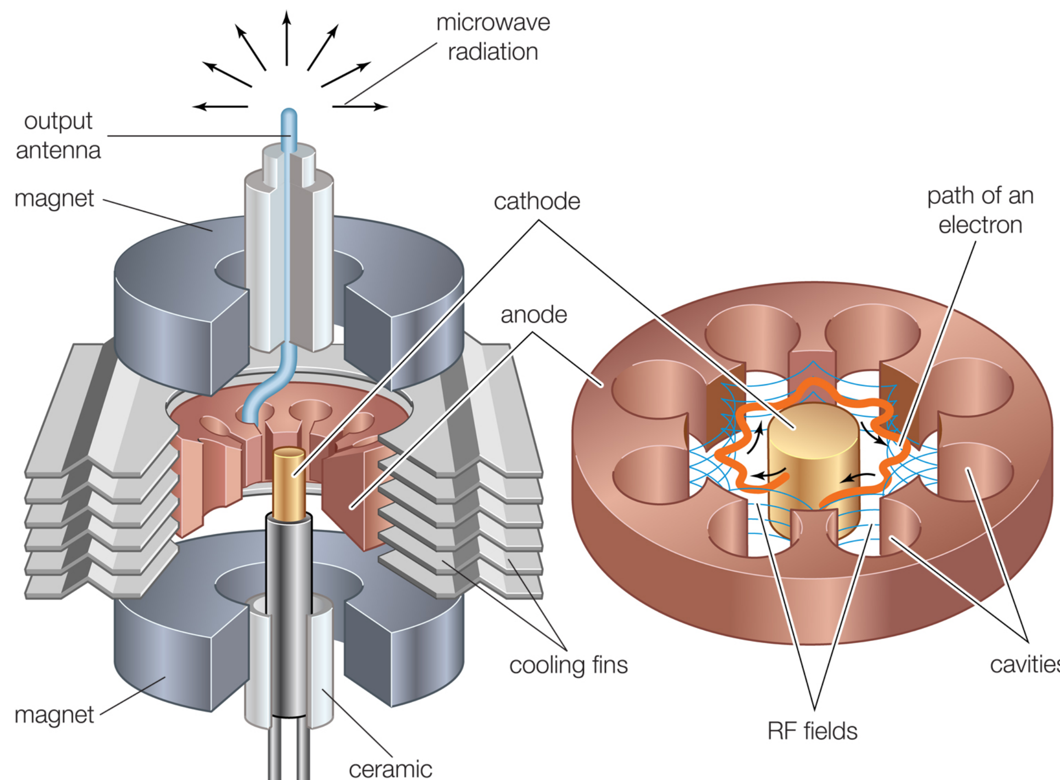


not resonant

Example of resonator: cavity magnetron source of 2.45 GHz radiation in microwave ovens (and WW2 radar)



$$\bar{F} = \underset{\uparrow \atop -e}{q}\,\bar{v} \times \bar{B}$$

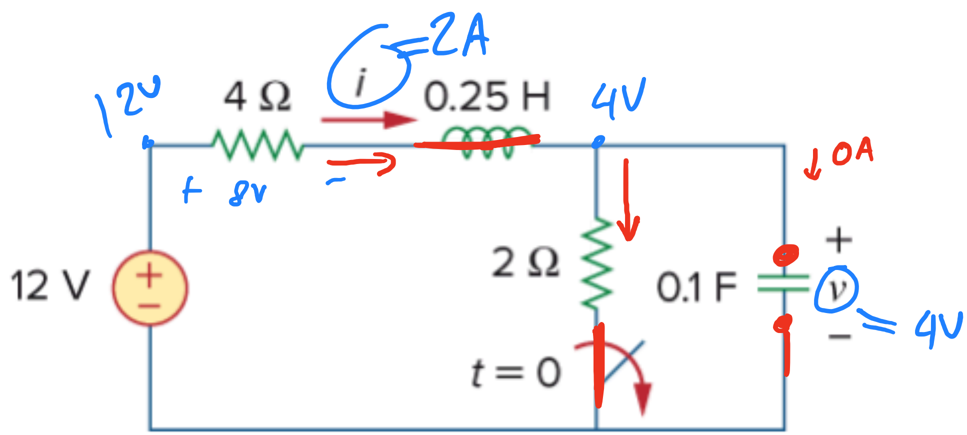


Initial conditions

- We will need: $v(0)$, $i(0)$, $dv(0)/dt$, $di(0)/dt$, $i(\infty)$, and $v(\infty)$.
- Remember: use passive sign convention
- For capacitor: $v(0^-) = v(0^+)$
- For inductor: $i(0^-) = i(0^+)$

Example 8.1

The switch in Fig. 8.2 has been closed for a long time. It is open at $t = 0$. Find:
(a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

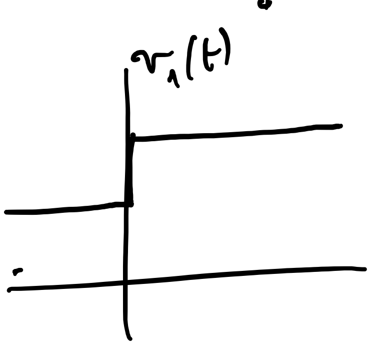


$$\frac{d\,i(0^+)}{dt} = \left.\frac{d\,i(t)}{dt}\right|_{t=0^+}$$

$t=0^-$: $i = 2A$, $v = 4V$, $\frac{di}{dt} = 0$, $\frac{dv}{dt} = 0$

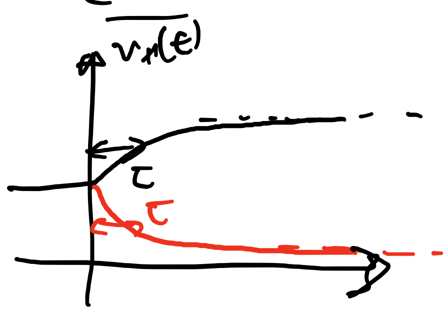
$t=0^+$: $i = 2A$, $v = 4V$

Only R's



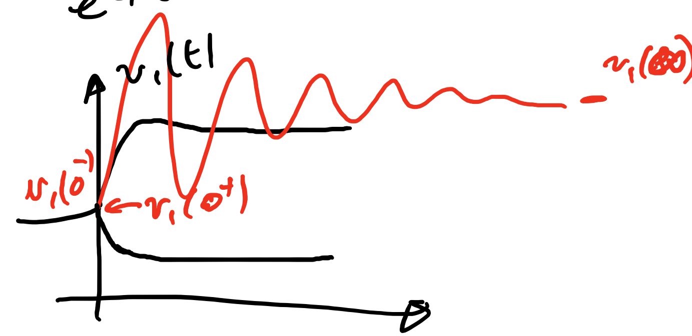
1st order

$e^{-t/\tau}$



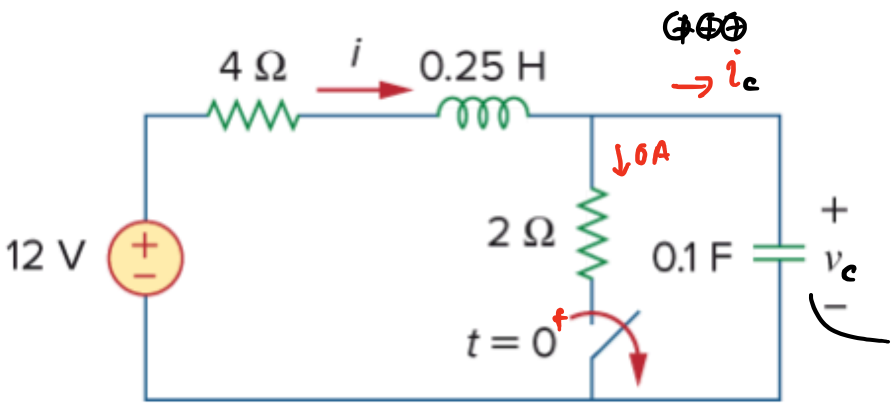
2nd order

$e^{-t/\tau}$



Example 8.1

The switch in **Fig. 8.2** has been closed for a long time. It is open at $t = 0$. Find:
(a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

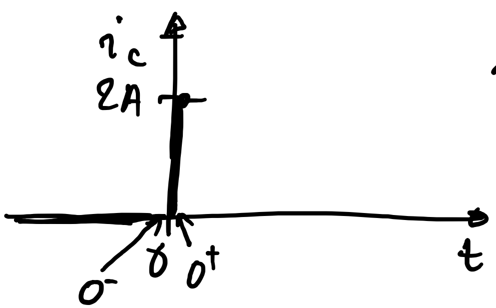


$Q = CV$

$$\frac{d\,i(0^+)}{dt} = \frac{di(t)}{dt}\Big|_{t=0^+}$$

$t=0^-$: $i = 2A$, $v = 4V$, $\frac{di}{dt} = 0$, $\frac{dv}{dt} = 0$

$t=0^+$: $i = 2A$, $v = 4V$



$$i_c(t=0) = C\,\frac{dv}{dt}\Big|_{t=0} \rightarrow \frac{dv_c}{dt}\Big|_{t=0} = \frac{i_c(0)}{C} = \frac{2A}{0.1F} = 20\ \text{V/s}$$

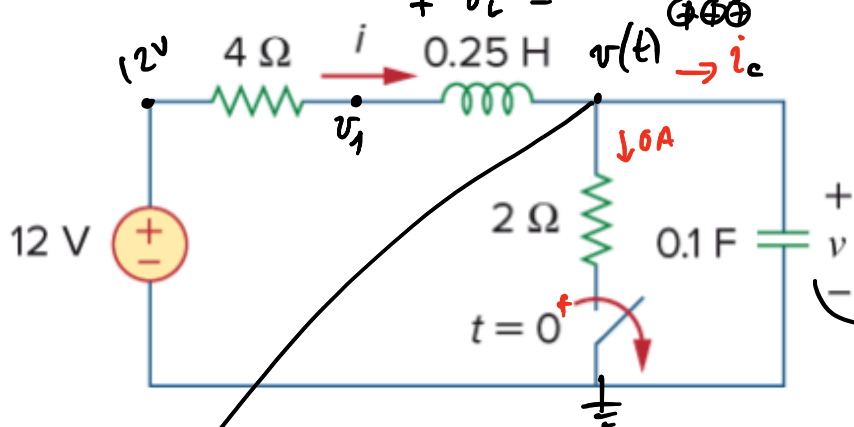
$\underline{2A}$

$$\frac{di}{dt}$$

Example 8.1

The switch in **Fig. 8.2** has been closed for a long time. It is open at $t = 0$. Find:
(a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

$i(0^+) = 2A$



$Q = CV$

$v(0^+) = 4V$

$t<0, v_L = 0$

$$\frac{d\,i(0^+)}{dt} \equiv \left.\frac{d\,i(t)}{dt}\right|_{t=0^+}$$

$t = 0^-$: $i = 2A$, $v = 4V$, $\frac{di}{dt} = 0$, $\frac{dv}{dt} = 0$

$t = 0^+$: $i = 2A$, $v = 4V$

$t<0$: $v_L = L\cancel{\frac{di}{dt}}^{\to 0} = 0$

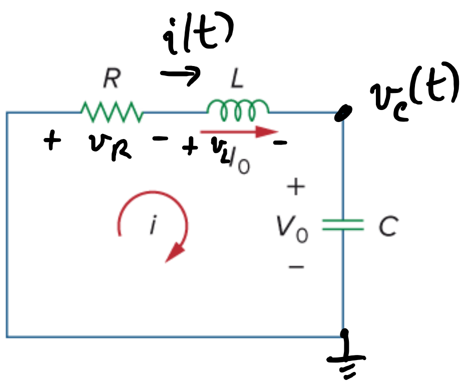
$t = 0^+$: $v_1 = 12V - 4\Omega \cdot 2A = 4V$, $v_L = L\left.\frac{di}{dt}\right|_{t=0^+}$ $\qquad \left.\frac{di}{dt}\right|_{t=0^+} = \frac{v_L}{L} = 0$

$= v_1 - v(t=0^+)$

$= 4V - 4V$

Source-free Series RLC Circuit

Give: $v(0)$, $i(0)$ at $t=0$. Find: $v(t)$, $i(t)$, $t>0$.



$$i = C\frac{dv_c}{dt} \longrightarrow \boxed{\frac{dv_c(t)}{dt} = \frac{i(t)}{C}} \Rightarrow v_c(t) = \int^{t} \frac{1}{C}\, i(t')\,dt'$$

$$v_R(t) + v_L(t) + v_c(t) = 0$$

$$iR + L\frac{di}{dt} + \frac{1}{C}\int^{t} i(t')\,dt' = 0$$

$\frac{d}{dt}$

$$\frac{dv_R}{dt} + \frac{d}{dt}v_L + \frac{d}{dt}v_c = 0$$

$$\boxed{R\frac{di}{dt} + L\frac{d^2}{dt^2}i + \frac{1}{C}i = 0} \rightarrow \text{need } i(0)$$

$$i'(0) = \frac{v_L(0)}{L} \Rightarrow v_c \cdots$$

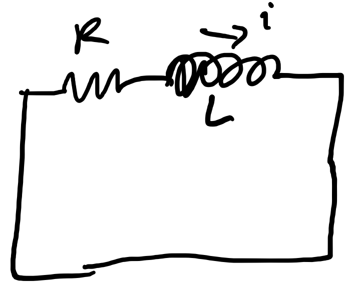
$$\boxed{\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0}$$

$$\frac{d^2}{dt^2}i + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

$\frac{1}{\tau_{RL}}$

$s^{-2} \rightarrow \frac{1}{\sqrt{LC}} \rightarrow s^{-1}$

$$\boxed{L\frac{d}{dt}i + iR = 0}$$



$$\frac{d}{dt}i + \frac{R}{L}i = 0$$

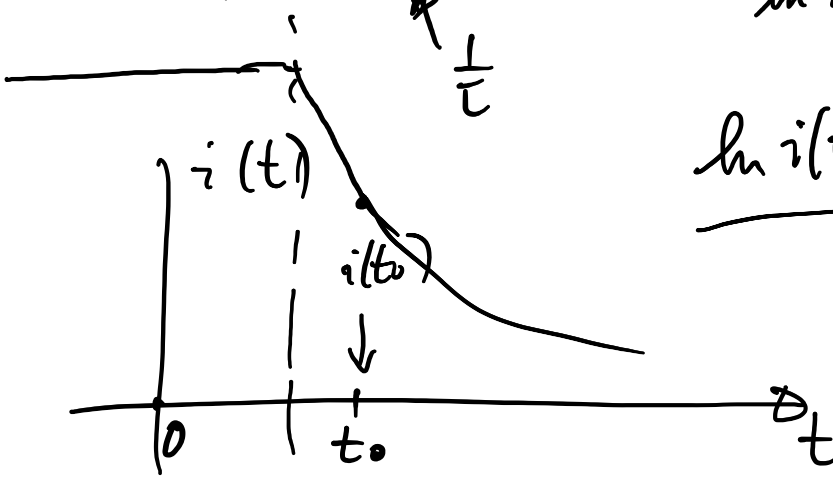
$$\frac{1}{i}\frac{d}{dt}i = -\frac{R}{L}$$

$$\frac{d}{dt}\ln i = -\frac{R}{L}$$

$$\ln i = -\frac{R}{L}\int dt = -\frac{R}{L}t$$

$$i(t) = e^{-\frac{R}{L}t}$$

$\frac{1}{L}$



$$\int_0^{\boxed{t}} \frac{d}{dt'}\ln i(t')\,dt' =$$

$$\ln i(t')\Big|_{t'=0}^{t'=t} = -\Big|_{t'=0}^{t'=t} = -\frac{R}{L}(t-0)$$

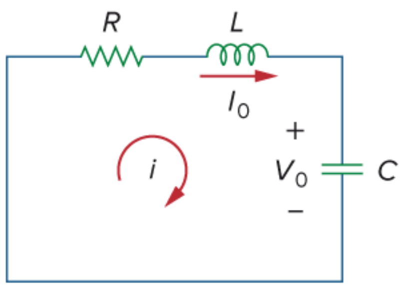
$$\ln i(t) - \ln i(0) = -\frac{R}{L}t \Rightarrow \frac{i(t)}{i(0)} = e^{-\frac{R}{L}t}$$

$$\int_{t_0}^{t}$$

$$\ln i(t) - \ln i(t_0) = -\frac{R}{L}(t-t_0)$$

$$\boxed{\frac{i(t)}{i(t_0)} = e^{-\frac{R}{L}(t-t_0)}}$$

Source-free Series RLC Circuit



$$\frac{d^2}{dt^2} i + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

$$s^2 \cancel{Ae^{st}} + \frac{R}{L} s \cancel{Ae^{st}} + \frac{1}{LC} \cancel{Ae^{st}} = 0$$

$$\boxed{s^2 + \frac{R}{L} s + \frac{1}{LC} = 0}$$

$s = -\frac{1}{\tau}$

$e^{-\frac{1}{\tau}t}$

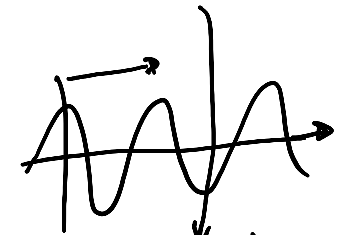
$$\boxed{i = Ae^{st}} \rightarrow \frac{di}{dt} = sAe^{st}, \quad \frac{d^2 i}{dt^2} = s^2 A e^{st}$$

$$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$\cos(2\pi f_0 t)$ Hz



$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

freq (rad/s)

$s = j\omega_0$

$e^{st} \Rightarrow e^{(j\omega_0)t}$

s_1 and s_2 : *natural frequencies*,
measured in nepers per second (Np/s).

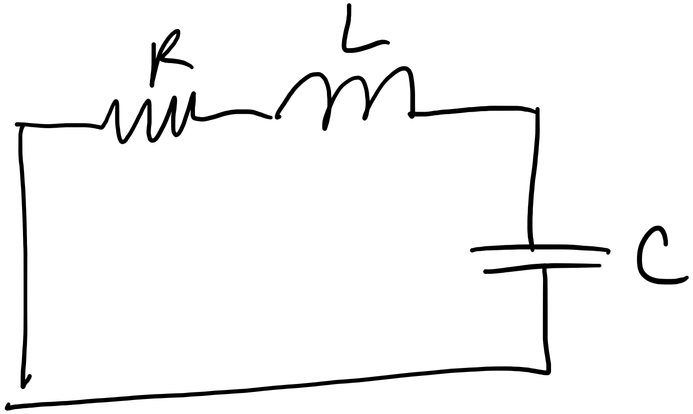
ω_0 : *resonant frequency* or strictly as the *undamped natural frequency*,
expressed in radians per second (rad/s)

$Ax^2 + Bx + C = 0$

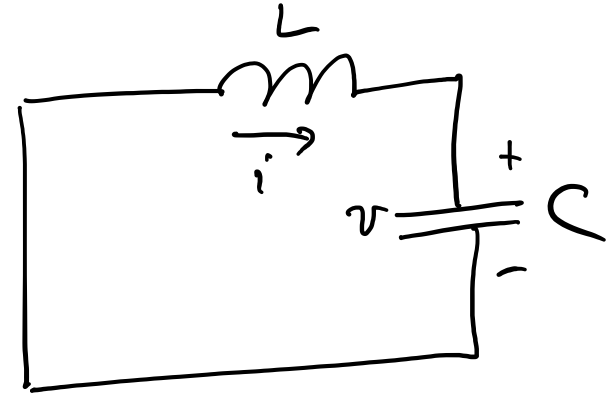
$x = \frac{-B \pm \sqrt{}}{2A}$

$$s = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4\frac{1}{LC}}}{2}$$

$$\boxed{s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}}$$



$R=0$ $\Rightarrow$



$$s = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

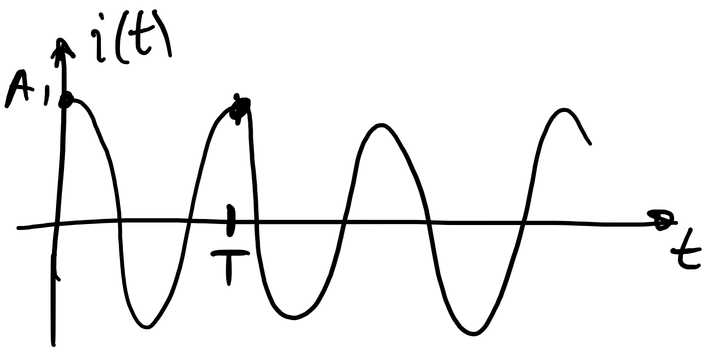
$$s = 0 \pm \sqrt{0 - \frac{1}{LC}}$$

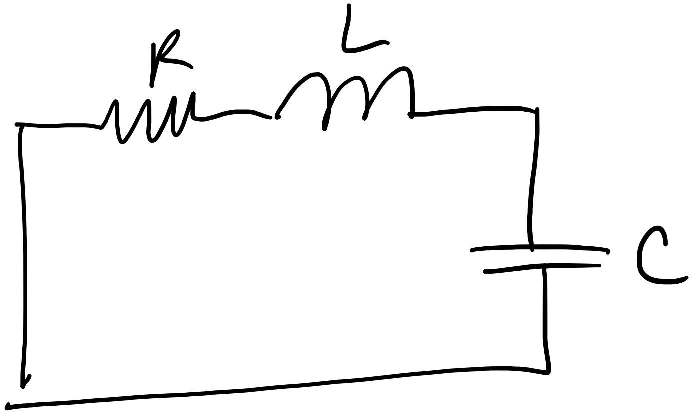
$$s_{1,2} = \pm j \frac{1}{\sqrt{LC}} = \pm j\omega_0$$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

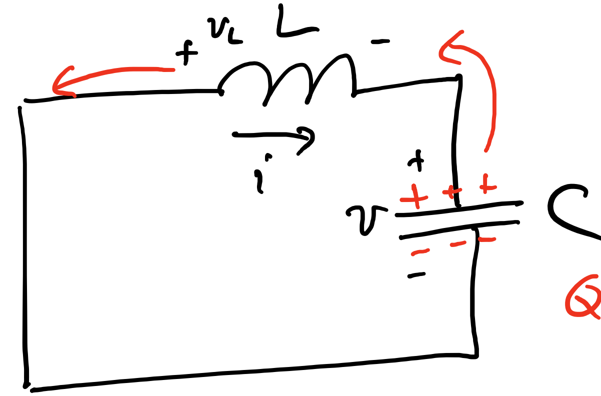
$$= A_1\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \underline{A_1 2\cos(\omega_0 t)}$$

$$T = \frac{2\pi}{\omega_0}$$

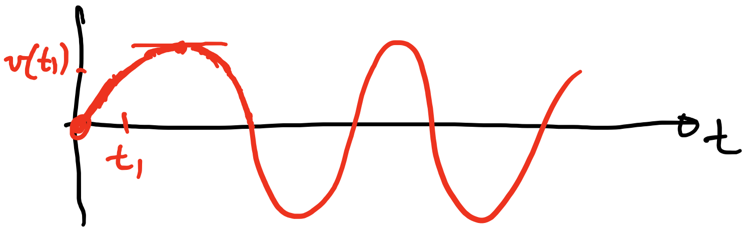




$R=0$



$Q = CV$

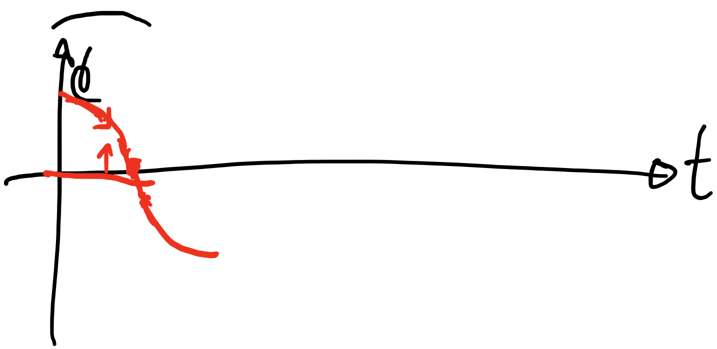


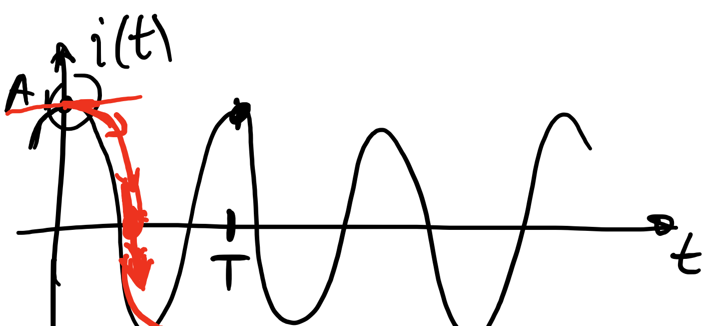
$$-v = +v_L = L\frac{di}{dt} \qquad i = C\frac{dv}{dt}$$

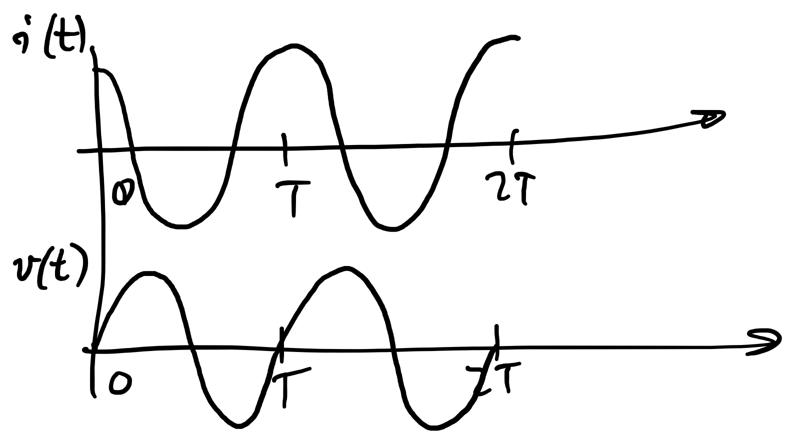
$$\frac{di}{dt} = \frac{-v}{L}$$

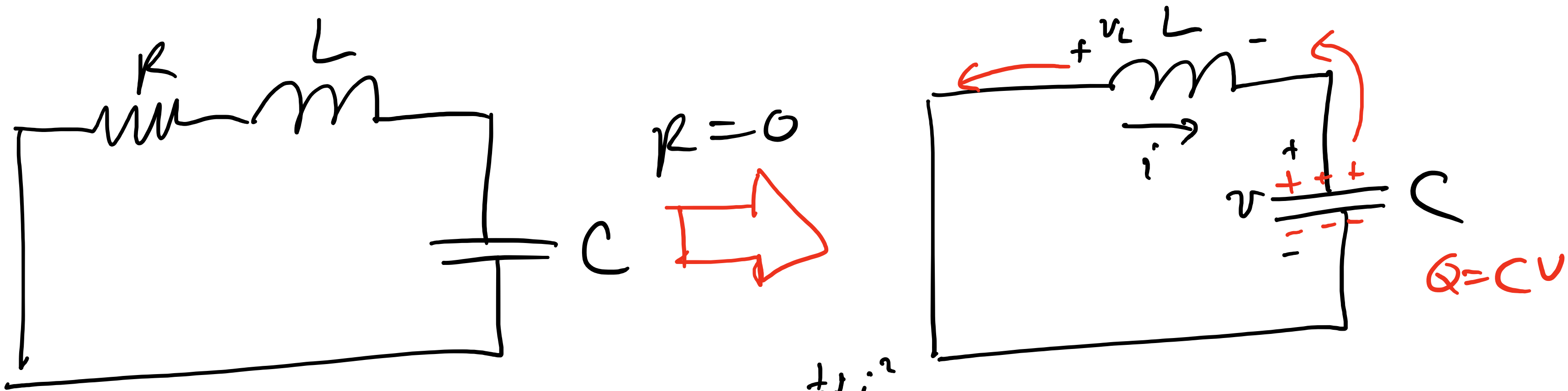
$$= 0 \text{ at } t=0$$

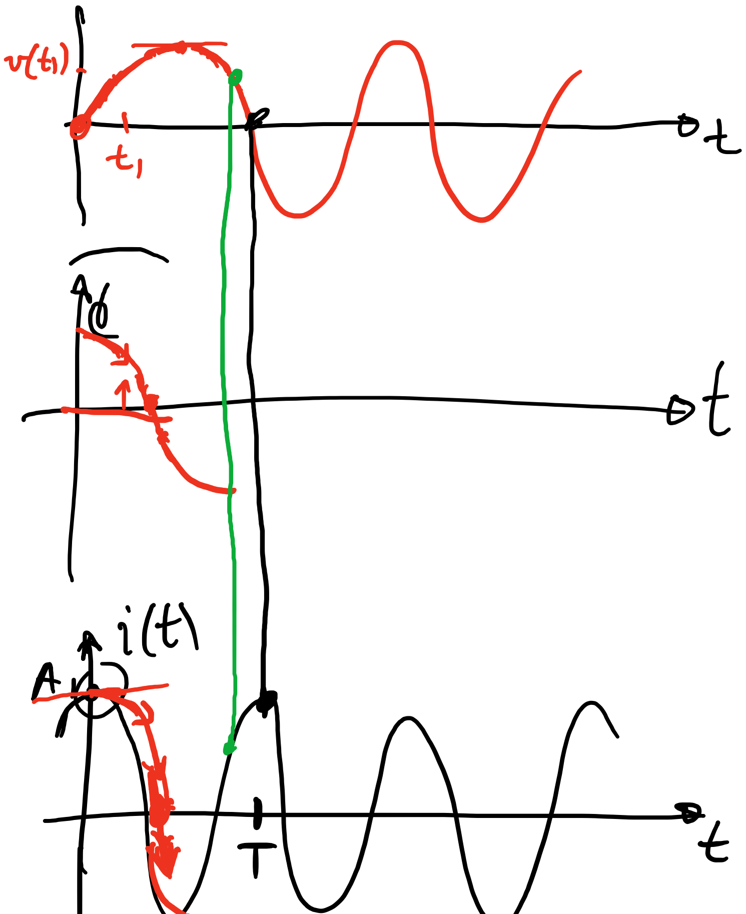
$$\left.\frac{di}{dt}\right|_{t=t_1} = \frac{-v(t_1)}{L} < 0$$

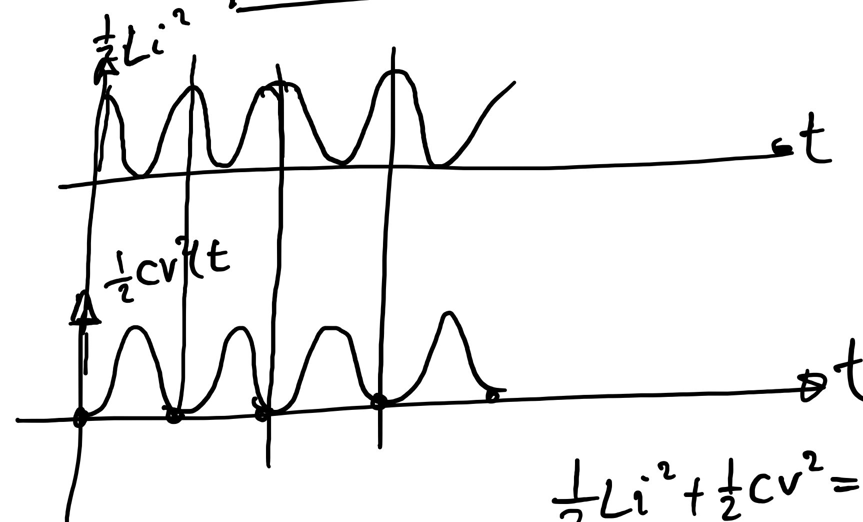




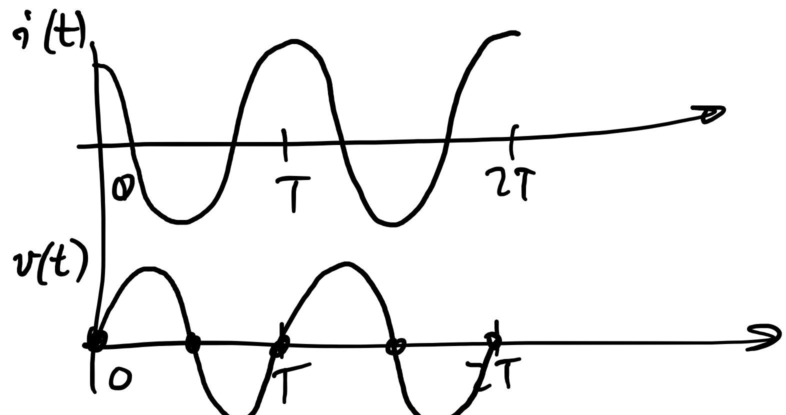




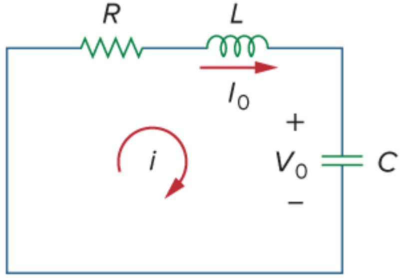




$$\frac{1}{2}Li^2 + \frac{1}{2}Cv^2 = \text{const.}$$



Source-free Series RLC Circuit



$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t}$$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

where

$$\boxed{\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}}$$

resonant freq.

We can infer 3 types of soln:

1) If $\alpha > \omega_0$, we have the *overdamped* case.
2) If $\alpha = \omega_0$, we have the *critically damped* case.
3) If $\alpha < \omega_0$, we have the *underdamped* case.

Source-free Series RLC Circuit

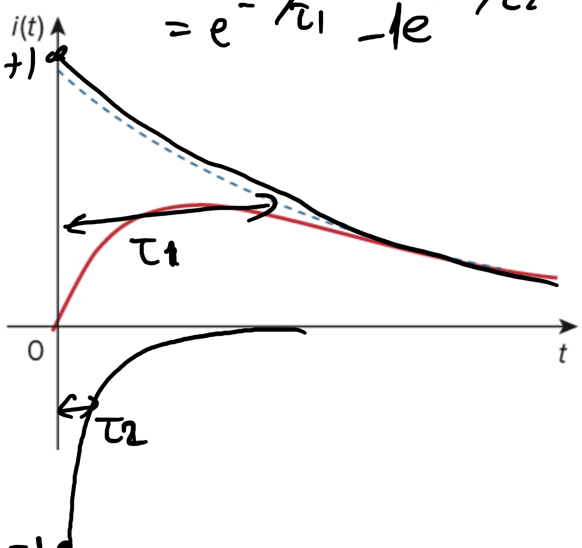
We can infer 3 types of soln:

1) If $\alpha > \omega_0$, we have the *overdamped* case.
2) If $\alpha = \omega_0$, we have the *critically damped* case.
3) If $\alpha < \omega_0$, we have the *underdamped* case.

$s = -\alpha \pm \Delta\alpha$

$i(t) = e^{-\alpha_1 t} - e^{-\alpha_2 t}$

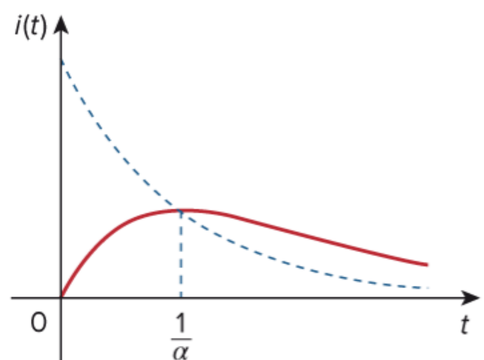
$= e^{-t/\tau_1} - 1e^{-t/\tau_2}$



Overdamped

$s = -\alpha \pm 0$

$i(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$



Critically damped

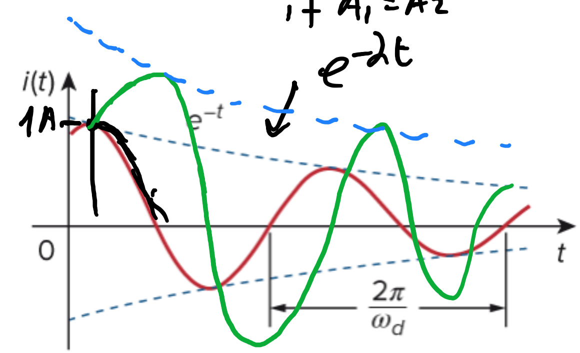
$s = -\alpha \pm j\omega_0$

$i(t) = A_1 e^{(-\alpha + j\omega_0)t} + A_2 e^{(-\alpha - j\omega_0)t}$

$= A_1 \, 2\cos(\omega_0 t) \, e^{-\alpha t}$

if $A_1 = A_2$

$e^{-\alpha t}$



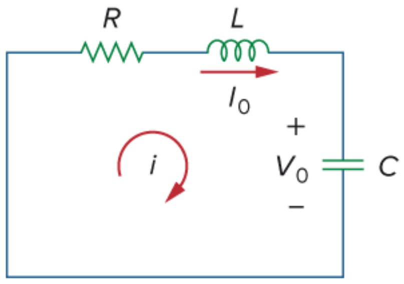
$i(0) = 1A, \; v(0) = 0$

$i(0) = 1A, \; v(0) = 1V$

Underdamped

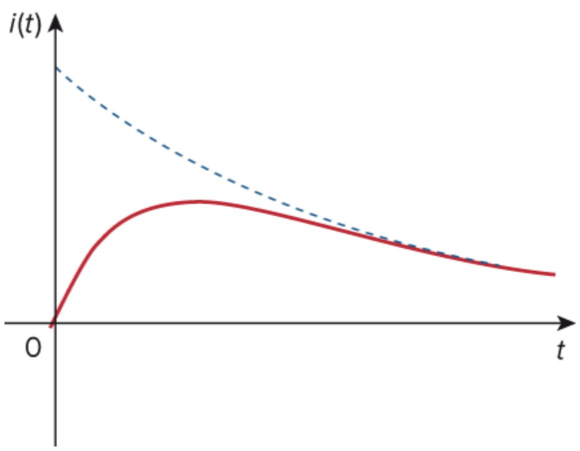
Overdamped case

$\alpha > \omega_0$ ⇒ $C > 4L/R^2$



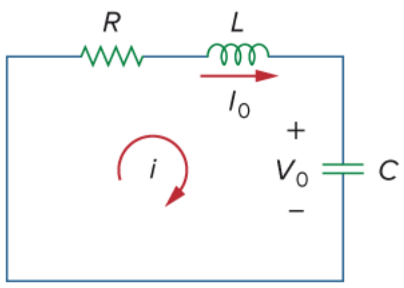
$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$



Critically damped case

$\alpha = \omega_0 \Rightarrow C = 4L/R^2$



$$s_1 = s_2 = -\alpha = -\frac{R}{2L}$$

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

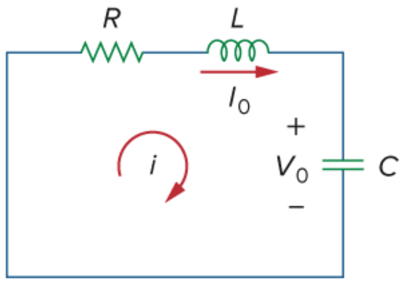
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Assumption of exponential solution wrong for critical damping case!

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \Rightarrow \frac{d^2 i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0$$

Critically damped case

$\alpha = \omega_0 \Rightarrow C = 4L/R^2$



$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\frac{d^2 i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0$$

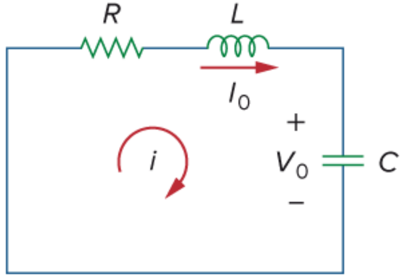
$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0$$

$$f = \frac{di}{dt} + \alpha i$$

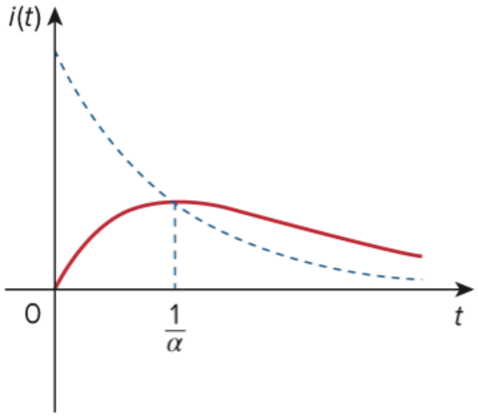
$$\frac{df}{dt} + \alpha f = 0$$

Critically damped case

$\alpha = \omega_0 \Rightarrow C = 4L/R^2$



$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$



$$\frac{df}{dt} + \alpha f = 0$$

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

$$\frac{d}{dt}(e^{\alpha t} i) = A_1$$

$$e^{\alpha t} i = A_1 t + A_2$$

$$i = (A_1 t + A_2)e^{-\alpha t}$$