9.2 SINUSOIDAL STEADY-STATE AMPLIFIER RESPONSE

The various capacitances described in Section 9.1, as well as any discrete capacitors specifically added by the designer, all influence the response of an electronic circuit. Indeed, the body of this chapter deals with methods for dealing with and predicting the effect of capacitance on circuit response. Before embarking on a study of these effects, however, we first review several key concepts and definitions that pertain to the *frequency domain*. In the frequency domain, a circuit is assumed to have been excited for some time by a sinusoidal input, such that all natural, transient responses have decayed to zero. Under such *sinusoidal steady-state* excitation, every voltage and current signal in the circuit acquires the frequency of the input and can be represented by a phasor. More importantly, each capacitor in the circuit can be represented by a frequency-dependent impedance of value $1/j\omega C$. This feature transforms the differential equations that normally govern capacitive circuits into simple algebraic equations. Any arbitrary input signal can always be represented as a Fourier-series superposition of sinusoids of different frequencies and amplitudes. Knowledge of the circuit's response to the individual sinusoidal Fourier components of the input allows the designer to predict the circuit's response to a complex periodic signal. The next three sections are devoted to a review of concepts that are important to the frequency domain. The study of actual circuits that contain capacitance begins in Section 9.3.

9.2.1 Bode Plot Representation in the Frequency Domain

The input–output response of a circuit in the frequency domain under sinusoidal steady-state conditions is called the circuit's *system function*, or sometimes the *transfer function*.[3] The system function contains a wealth of information about the circuit's steady-state behavior under sinusoidal excitation. This information is neatly expressed in the compact, graphical form of a *Bode plot* (pronounced "Bo-dee"). When a linear circuit has a frequency-dependent system function, both the magnitude and phase angle of the response are variables of great interest. It is often useful to know their values over very large ranges in frequency spanning several orders of magnitude. Similarly, it is often desirable to assign equal importance to the lower and higher ends of the frequency spectrum. The Bode plot consists of a set of straight lines placed on a graph with the frequency on the horizontal axis and either the output amplitude or phase angle on the vertical axis. The straight lines serve as asymptotes that closely represent the actual circuit response, but are much easier to manipulate and analyze. We shall first develop the Bode plot for the simple circuits of Figs. 9.10 and 9.11. These simple circuits highlight the key role of capacitors in many electronic circuits. We then extend the concept to encompass more complicated circuits having system functions of arbitrary complexity.

3 More accurately, the term *transfer function* is used to describe the frequency-domain relationship between input and output signals appearing in different parts of the circuit. The more general term *system function* includes transfer functions, but can also be used to describe the impedance or admittance of a single port.

Solution

$H(j\omega)$ begins at low frequencies with a solitary factor of $j\omega$ and an initial slope of $+20$ dB/decade. The flat midband region thus begins at the lowest-frequency pole $\omega = 10$ rad/s. The upper -3-dB limit of the midband region can be estimated by superimposing the two remaining high-frequency poles:

$$\omega_H = \omega_1 \| \omega_2 = \frac{1}{1/10^4 + 1/(2 \times 10^4)} \approx 0.67 \times 10^4 \text{ rad/s} \tag{9.73}$$

This estimated value for ω_H is slightly lower than the true value $\omega_H = 0.84 \times 10^4$ rad/s obtained from Eq. (9.58).

(ERCISE 9.6 Find ω_L, ω_H, and the midband gain of the system function

$$(j\omega) = 5 \frac{j\omega/2}{(1 + j\omega/2)(1 + j\omega/10^5)(1 + j\omega/10^6)}$$

Answer: $\omega_L = 2$ rad/s; $\omega_H \approx 9.1 \times 10^4$ rad/s; $A_o = 5 \equiv 14$ dB

9.7 Find ω_L, ω_H, and the midband gain of the system function of Exercise 9.5.
Answer: $\omega_L = 2.3 \times 10^4$ rad/s; $\omega_H = 10^6$ rad/s; $A_o = 1.1 \times 10^5 \equiv 101$ dB

.3 FREQUENCY RESPONSE OF CIRCUITS CONTAINING APACITORS

The concepts presented in Section 9.2 provide powerful tools for working in the frequency domain. With these tools mastered we can now understand the effects of capacitance (and inductance, where important) on circuit behavior. In the sections that follow, we shall use these tools to analyze and design real electronic circuits. To facilitate the connection between the abstract concepts of Section 9.2 and the real circuits of the rest of the chapter, we first provide several key definitions that help categorize the role of each capacitor in shaping circuit response.

3.1 High- and Low-Frequency Capacitors

The influence of a given capacitance often occurs at a frequency that lies either above or below a circuit's midband region. Conversely, the midband represents the frequency range over which circuit behavior is unaffected by circuit capacitance. From a frequency-domain point of view, it is often useful to categorize a given capacitor as either a *high-frequency* or *low-frequency* type, depending on whether its effects are felt above or below the midband range. In an amplifier, a high-frequency capacitor is defined as one that degrades the gain above the midband range. Similarly, a low-frequency capacitor is defined as one that degrades the gain below the midband range. Because capacitive impedance is inversely proportional to frequency, it follows that a low-frequency capacitor must behave as a short circuit in the midband, while a high-frequency capacitor must behave as an open circuit in the midband.

As a general rule, a given capacitor will function as a low-frequency type if it appears in series with a circuit's input or output terminal. Conversely, a capacitor will function as a high-frequency type if it shunts an input or output node to small-signal ground. According to this

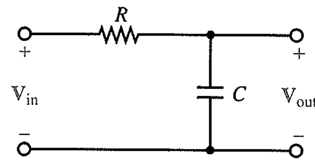


Figure 9.10
Simple RC circuit with the capacitor as a shunt element.

In general, the use of Bode plots is limited to linear circuits. Many nonlinear circuits, however, including the amplifier circuits of this chapter, can be represented by frequency-dependent piecewise-linear or small-signal circuit models. The Bode-plot formulation is useful for describing the small-signal frequency response of these circuits as well.

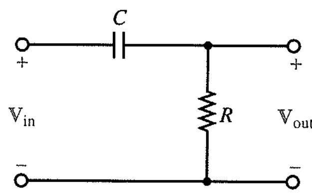


Figure 9.11
Simple RC circuit with the capacitor as a series element.

A complete Bode plot consists of two separate parts. The first shows the magnitude of the output variable relative to the input variable as a function of frequency. The second part shows the phase angle of the output variable relative to the input variable as a function of frequency. The angle of the input variable is arbitrarily (and for convenience) taken as the zero-angle reference.

As an example, consider the Bode plot for the simple circuit of Fig. 9.10, which consists of a series resistor and a *shunt*, or parallel, capacitor. The system function of this circuit becomes, via voltage division

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC} \tag{9.32}$$

where the capacitor is treated as an element having impedance $1/j\omega C$. As an aid in drawing the Bode plot, we note the behavior of the system function at three extremes of frequency. In the low-frequency limit $\omega \ll 1/RC$, the imaginary part of the denominator becomes negligible, and the system function (9.32) reduces to $\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}} = 1$ so that

$$\left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right| = 1 \tag{9.33}$$

and

$$\measuredangle\, \mathbf{V}_{\text{out}} = 0 \tag{9.34}$$

where the angle of $\mathbf{V}_{\text{in}}$ is taken as the zero-angle reference.

In the high-frequency limit $\omega \gg 1/RC$, the imaginary term in the denominator of Eq. (9.32) becomes larger than the real term, so that the system function reduces to

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} \rightarrow \frac{1}{j\omega RC} \tag{9.35}$$

with

$$\left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right| = \frac{1}{\omega RC} \quad \text{and} \quad \measuredangle\, \mathbf{V}_{\text{out}} = -90^\circ \tag{9.36}$$

In this limit of large ω, the magnitude $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}|$ *decreases* by a factor of 10 for every factor-of-10 *increase* in ω.

At the boundary between high- and low-frequency extremes, which occurs at the point $\omega = 1/RC$, the magnitude of the real and imaginary terms of the denominator of Eq. (9.32) become equal to each other, so that the magnitude and angle of the system function become

$$\left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right| = \left|\frac{1}{1+j}\right| = \frac{1}{\sqrt{2}} = 0.707 \tag{9.37}$$

and

$$\measuredangle\, \mathbf{V}_{\text{out}} = -\measuredangle\,(1+j) = -45^\circ \tag{9.38}$$

In Fig. 9.12, the magnitude and phase angle of the circuit of Fig. 9.10 are plotted as functions of frequency on logarithmic scales. The plots include the three limiting region cases described above. Both magnitude and frequency are plotted logarithmically, so that the high and low ends of the axes are given equal graphical weighting. Note that a logarithmic scale has no zero point and a logarithmic graph has no origin; hence the point at which the vertical and horizontal axes cross on the magnitude plot is arbitrary.

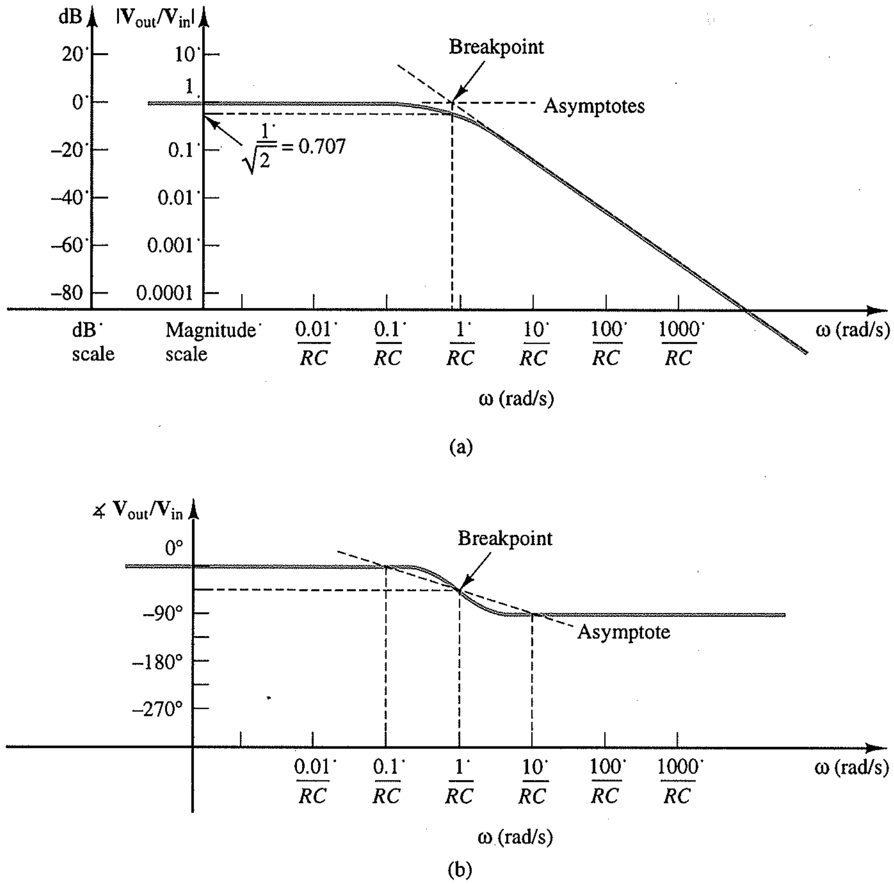


Figure 9.12 Plot of the frequency response of the circuit of Fig. 9.10: (a) magnitude plot; (b) phase-angle plot.

For $\omega \ll 1/RC$, the magnitude plot approaches the horizontal asymptote $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}| = 1$ given by Eq. (9.33). For $\omega \gg 1/RC$, the plot approaches the asymptote given by Eq. (9.36). These asymptotes together constitute the circuit's magnitude Bode plot. Above their point of intersection at $\omega = 1/RC$, the right-hand asymptote slopes downward by a factor of 10 for every factor-of-10 increase in ω. It can be shown that at the breakpoint $\omega = 1/RC$, the actual magnitude curve falls by $1/\sqrt{2}$ from the value at the point of intersection. The phase-angle plot has two horizontal asymptotes—one for $\omega \ll 1/RC$ and one for $\omega \gg 1/RC$—located at 0° and −90°, respectively. The phase-angle plot passes through the −45° point at the breakpoint $\omega = 1/RC$.

It is often convenient to express the logarithmic magnitude scale of the Bode plot with a unit called the magnitude *decibel*, defined by

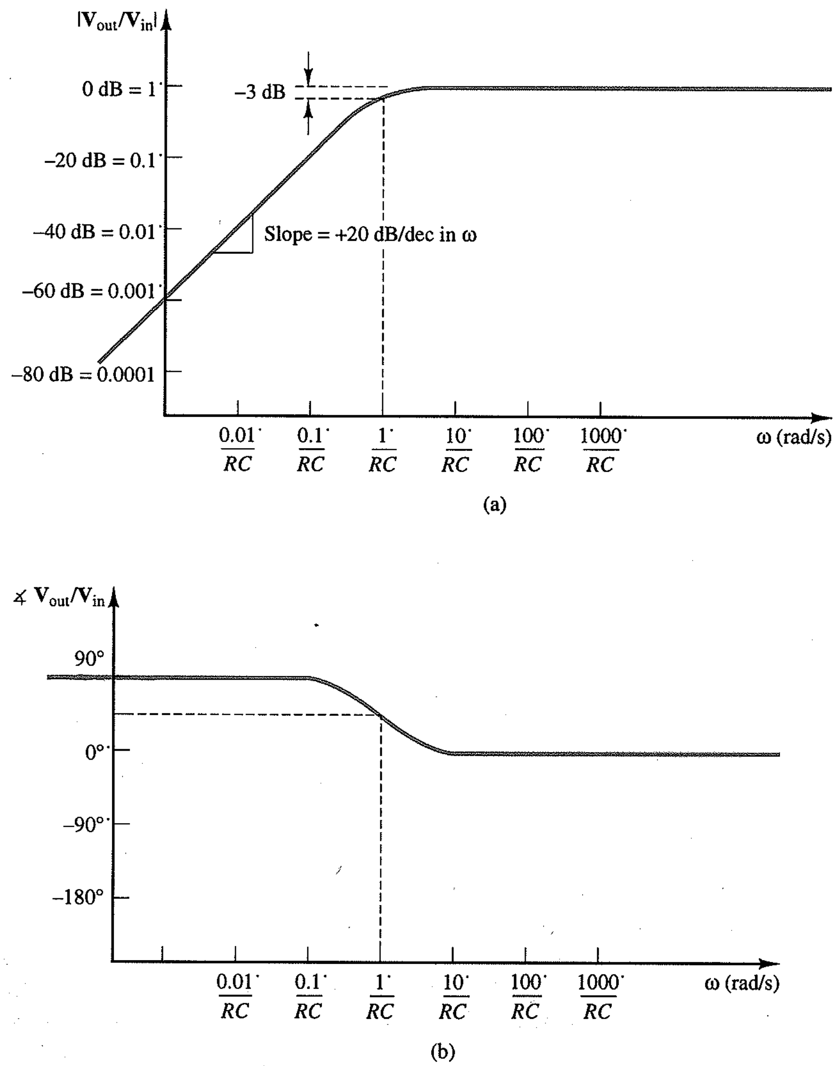
$$\text{dB} = 20 \log_{10} \left| \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} \right| \tag{9.39}$$

The decibel is a logarithmic unit; hence a dB scale used in a logarithmic plot appears linear, as in Fig. 9.12(a).

We next consider the circuit of Fig. 9.11, which consists of a series capacitor and a shunt resistor. The system function of this circuit is given, again using voltage division, by

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{R}{R + 1/j\omega C} = \frac{j\omega RC}{1 + j\omega RC} \tag{9.40}$$

Figure 9.13 Plot of the frequency response of the circuit of Fig. 9.11: (a) magnitude plot: (b) phase-angle plot.



The Bode-plot asymptotes can be found from the three limits at $\omega \ll 1/RC$, $\omega \gg 1/RC$, and $\omega = 1/RC$:

$$\text{At} \quad \omega \ll 1/RC: \left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right| \to \left|\frac{j\omega RC}{1}\right| = \omega RC \qquad \text{and } \measuredangle\, \mathbf{V}_{\text{out}} \to +90^\circ \tag{9.41}$$

$$\text{At} \quad \omega \gg 1/RC: \left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right| \to \left|\frac{j\omega RC}{j\omega RC}\right| = 1 \qquad \text{and } \measuredangle\, \mathbf{V}_{\text{out}} \to 0^\circ \tag{9.42}$$

$$\text{At} \quad \omega = 1/RC: \left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right| = \left|\frac{j}{1+j}\right| = \frac{1}{\sqrt{2}} \qquad \text{and } \measuredangle\, \mathbf{V}_{\text{out}} = 90^\circ - 45^\circ = 45^\circ \tag{9.43}$$

The low-frequency limit (9.41) has a factor of ω in the numerator. For $\omega \ll 1/RC$, the magnitude plot thus approaches an asymptote with an upward slope of 20 dB per factor-of-10 change in ω, as shown in Fig. 9.13. Similarly, for $\omega \gg 1/RC$, the magnitude plot asymptotically approaches a constant value of unity. It can be shown for this system function that the low- and high-frequency asymptotes cross at the breakpoint $\omega = 1/RC$, where the actual magnitude plot passes $1/\sqrt{2}$ below the breakpoint crossing. The factor of $1/\sqrt{2} = 0.707$ can also be expressed in decibels as

$$\text{dB} = 20 \log_{10} 0.707 \approx -3\,\text{dB} \tag{9.44}$$

At low frequencies, the Bode plot of Fig. 9.13 has an upward slope of +20 dB per decade in ω. This slope results because the low-frequency limit (9.41) has a factor of ω in the numerator. Suppose, for example, that at some low frequency $\omega_1 \ll 1/RC$, the magnitude has a decibel value of

$$\text{dB}_1 = 20 \log_{10} |\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}| = 20 \log_{10} \omega_1 RC \tag{9.45}$$

where $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}|$ is expressed using the limiting case (9.41). At some higher frequency $\omega_2 = 10\omega_1$ that still satisfies the limit $\omega_2 \ll 1/RC$, the decibel value becomes

$$\text{dB}_2 = 20 \log_{10} 10\omega_1 RC = 20 \log_{10} \omega_1 RC + 20 \log_{10} 10 = \text{dB}_1 + 20 \tag{9.46}$$

This value is 20 decibels more than the decibel value at ω_1.

EXERCISE 9.1 Draw the magnitude and angle Bode plots for the circuits of Figs. 9.10 and 9.11 if the capacitor is replaced by an *inductor* of value L.

9.2 Show that the slopes of the nonhorizontal portions of the magnitude plots of Figs. 9.12 and 9.13 have values equal to ±6 dB per *octave*, where an octave is a factor-of-2 change in frequency.

9.2.2 Bode-Plot Representation of System Functions of Arbitrary Complexity

In later sections of this chapter, we will examine circuits with system functions that are far more complex than those of Eqs. (9.32) and (9.40). Fortunately, the task of constructing the Bode plot of any circuit, no matter how complex, is greatly simplified if its system function can be expressed in the general form

$$H(j\omega) = A \frac{j\omega(1 + j\omega/\omega_2)(1 + j\omega/\omega_4)\cdots}{(1 + j\omega/\omega_1)(1 + j\omega/\omega_3)(1 + j\omega/\omega_5)\cdots} \tag{9.47}$$

The numbered frequencies $\omega_1 \cdots \omega_n$ are the *breakpoints* of the system function, and A is a constant. The solitary factor of $j\omega$ in the numerator is not present for all circuits. If the binomial containing a given breakpoint frequency ω_n appears in the numerator, then ω_n is called a *zero* of the system function. If the binomial appears in the denominator, then ω_n is called a *pole*.

Regardless of its type, a binomial term containing ω_n will affect the circuit's magnitude and phase response as the driving frequency approaches and passes through the value ω_n.

Suppose that the frequency ω of the input signal driving the circuit initially lies well below ω_n. In such a case, the binomial term containing ω_n will alter neither the magnitude nor the phase of the system function, but will simply multiply the system function by unity. This statement can be verified by observing the characteristics of a single binomial term for frequencies well below ω_n:

$$\left|1+\frac{j\omega}{\omega_n}\right| \approx 1 \qquad \text{for} \qquad \omega \ll \omega_n \tag{9.48}$$

and

$$\measuredangle\left(1+\frac{j\omega}{\omega_n}\right) \approx 0^\circ \tag{9.49}$$

Conversely, if the driving frequency ω lies well *above* a given breakpoint frequency ω_n, the binomial term associated with ω_n will contribute a factor of ω/ω_n to the magnitude of the system function and an angle factor of 90°. This statement can be verified by noting that

$$\left|1+\frac{j\omega}{\omega_n}\right| \approx \frac{\omega}{\omega_n} \qquad \text{for} \qquad \omega \gg \omega_n \tag{9.50}$$

and

$$\measuredangle\left(1+\frac{j\omega}{\omega_n}\right) \approx \measuredangle\frac{j\omega}{\omega_n} = 90^\circ \tag{9.51}$$

If the binomial appears in the numerator as a zero, the factor of ω/ω_n will appear in the numerator, and the angle contribution of 90° will be *added* to the overall angle. If the binomial appears in the denominator as a pole, the contributed factor of ω/ω_n will appear in the denominator, and the angle contribution of 90° will be *subtracted* from the overall angle.

The transition between the extremes $\omega \ll \omega_n$ and $\omega \gg \omega_n$ occurs at $\omega = \omega_n$. At this frequency, the binomial of ω_n contributes a factor of $\sqrt{2}$ to the magnitude of the system function and an angle of 45°. The validity of this statement can be shown by noting that at $\omega = \omega_n$,

$$\left|1+\frac{j\omega}{\omega_n}\right| = |1+j| = \sqrt{1^2+1^2} = \sqrt{2} \tag{9.52}$$

and

$$\measuredangle\left(1+\frac{j\omega}{\omega_n}\right) = \measuredangle(1+j) = 45^\circ \tag{9.53}$$

When the numerator of the system function contains a single non-binomial factor of $j\omega$, a factor of ω will be contributed to the magnitude and a constant factor of 90° will be contributed to the phase angle at all values of the driving frequency ω.

Given these guidelines, the Bode-plot asymptotes that describe the magnitude and phase response of a system function of the form (9.47) are easily constructed. We briefly review the procedure here. The process begins by considering frequencies well below the lowest breakpoint of the system function. At such frequencies, the response will be flat (zero slope) with magnitude A and phase angle zero. (If the numerator contains a solitary factor of $j\omega$, the response at low frequencies will instead have a magnitude of $A\omega$, a phase angle of 90°, and an initial slope of +20 dB/decade.) The system function is next evaluated as the frequency is increased. As the frequency passes through a given breakpoint ω_n, its binomial term will begin to contribute a factor of ω/ω_n to the magnitude of the system function. If the binomial appears in the numerator, the slope of the asymptote describing the magnitude response will increase by +20 dB/decade. If the binomial term appears in the denominator, the slope of the asymptote will decrease by −20 dB/decade.

The angle portion of the Bode plot can be constructed in a similar fashion. When t binomial term appears in the numerator, the angle of the system function will undergo a pha shift of +90° as the frequency passes through the breakpoint. If the binomial term appears in t denominator, the phase shift will be −90°. The phase shift contributed *at* the breakpoint will i equal to +45° or −45°, respectively. If a solitary factor of $j\omega$ appears in the numerator of t system function, the Bode plot will *begin* with an upward slope of +20 dB/decade and a pha angle of +90° at low frequencies.

In the following examples, the techniques for constructing a Bode plot are illustrated f two cases. The first involves a system function whose response is flat at low frequencies. Th second involves a system function with a factor of $j\omega$ in the numerator.

EXAMPLE 9.1

Draw the magnitude and angle Bode plots of a circuit that has a frequency-domain system functic given by

$$H(j\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{100}{(1 + j\omega/10^2)(1 + j\omega/10^6)} \quad (9.5$$

Solution

The system function (9.54) has one pole at $\omega = 10^2$ rad/s and one at 10^6 rad/s. At frequencies we below the lowest pole at $\omega = 10^2$ rad/s, the magnitude of the system function is flat and approache the limit $|H| = 100 \equiv +40$ dB, as shown in Fig. 9.14. Above the pole at $\omega = 10^2$ rad/s, th asymptote describing the magnitude response acquires a slope of −20 dB/decade. The actua magnitude curve lies −3 dB below the asymptote intersection at point A. Above the second pol at $\omega = 10^6$ rad/s, the asymptote acquires an additional slope of −20 dB/decade, for a total slop of −40 dB/decade. With no other poles or zeros in the system function, this new slope continue indefinitely for all higher frequencies. The actual magnitude curve again lies −3 dB below th asymptote intersection at point B.

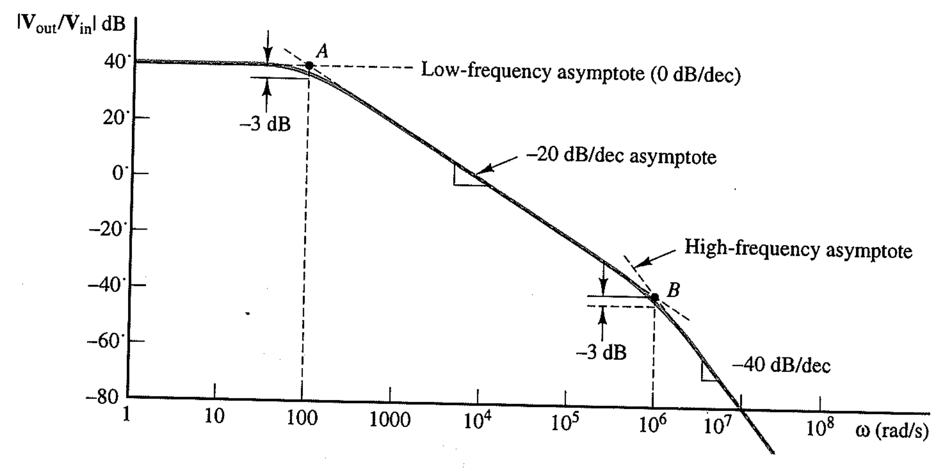


Figure 9.14 Magnitude plot of the system function of Eq. (9.54).

The angle plot of Eq. (9.54) is shown in Fig. 9.15. Well below $\omega = 10^2$ rad/s, the angle of the system function approaches zero. As the first pole at $\omega = 10^2$ rad/s is passed, the angle undergoes a net phase shift of −90°, with its value precisely at $\omega = 10^2$ rad/s equal to −45°. As the second pole at $\omega = 10^6$ rad/s is passed, it contributes an additional phase shift of −90°, for a total phase shift of −180° well above $\omega = 10^6$ rad/s. The total phase shift precisely at $\omega = 10^6$ rad/s is −135°, with −90° contributed from the pole at $\omega = 10^2$ rad/s and −45° contributed by the pole at $\omega = 10^6$ rad/s

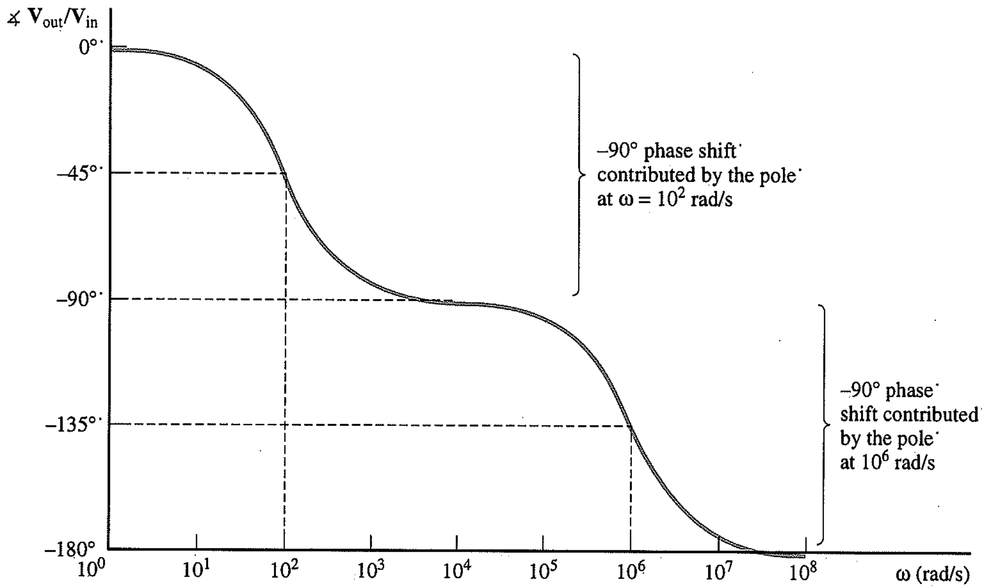


Figure 9.15 Angle plot of the system function of Eq. (9.54).

EXAMPLE 9.2 Construct the Bode plot of a circuit whose input–output system function is given by

$$H(j\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = 50\frac{j\omega(1 + j\omega/10)}{(1 + j\omega/10^4)(1 + j\omega/10^7)} \tag{9.55}$$

This system function has a solitary factor of $j\omega$ in the numerator.

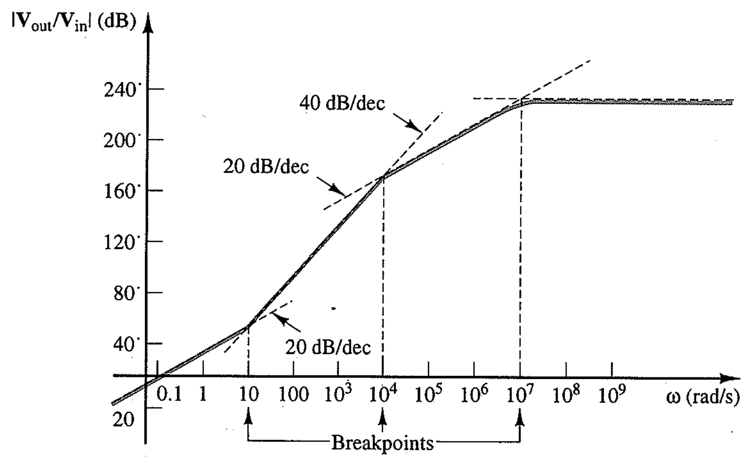


Figure 9.16 Magnitude plot of the system function of Eq. (9.55).

Solution

The magnitude Bode plot of $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ for the system function (9.55) is shown in Fig. 9.16. The point of intersection of the two axes is arbitrary. The system function contains a solitary factor of $j\omega$ in the numerator, hence the plot begins with a positive slope of +20 dB/decade for frequencies below the lowest breakpoint $\omega = 10$ rad/s. At the frequency $\omega = 10$, the zero in the numerator takes effect and the slope of the Bode-plot asymptote acquires an additional factor of +20 dB/decade to become +40 dB/decade. At the frequency $\omega = 10^4$ rad/s, the first pole in the denominator is encountered, and the asymptote slope is reduced by −20 dB/decade to again become +20 dB/decade. Finally, at the second pole frequency $\omega = 10^7$ rad/s, the asymptote acquires another factor of −20 dB/decade and becomes horizontal for all frequencies greater than

$\omega = 10^7$ rad/s, which is the highest breakpoint of the system function. At each of the breakpoints in the system function (9.55), the actual frequency-response curve falls +3 dB or −3 dB above or below the intersection points of the asymptotes.

Well above the highest breakpoint frequency $\omega = 10^7$, the magnitude of the system function can be approximated by

$$\frac{|\mathbf{V}_{out}|}{|\mathbf{V}_{in}|} \approx 50\frac{\omega(\omega/10)}{(\omega/10^4)(\omega/10^7)} = \frac{50(10^4)(10^7)}{10} = 5(10^{11}) \equiv 234\,\text{dB} \tag{9.56}$$

Note that the factors of ω^2 cancel out in the numerator and denominator in Eq. (9.56), leaving a term that is constant with frequency.

The angle portion of the Bode plot of Eq. (9.55) is shown in Fig. 9.17. In this case, the solitary factor of $j\omega$ in the numerator contributes an initial angle of +90° to the plot. Above the zero at $\omega = 10$ rad/s, an additional angle of +90° is contributed, making the total angle +180°. Above the next breakpoint at $\omega = 10^4$, which is a pole, the angle is reduced by −90° to +90°. Above the highest pole at $\omega = 10^7$, the total system function angle is again reduced by −90° to zero, which is a result consistent with the horizontal slope of the magnitude plot at high frequencies. Note that precisely at the location of each of the breakpoints, the system function angle is shifted by half the overall 90° angle shift contributed by the breakpoint.

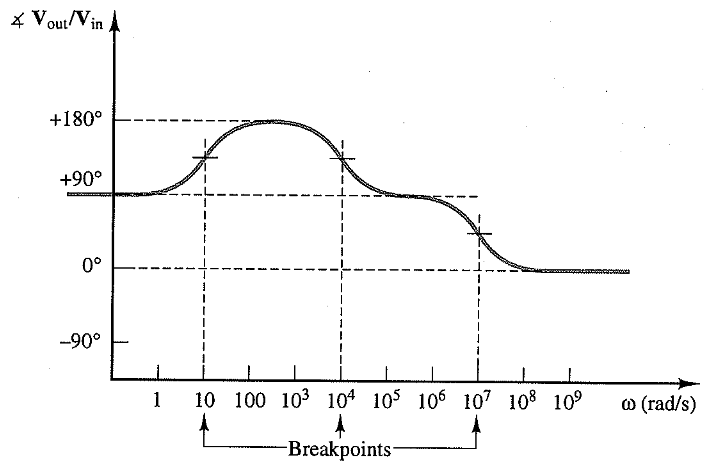


Figure 9.17 Angle plot of the system function of Eq. (9.55).

EXERCISE 9.3 Draw the magnitude and angle Bode plots of the circuits of Figs. 9.10 and 9.11 if $R = 5\,\text{k}\Omega$ and $C = 10\,\mu\text{F}$.

9.4 A circuit has a system function with poles at $\omega = 500$ rad/s and 3×10^5 rad/s. At $\omega = 0$, the system function has a value of 50. Draw its magnitude and angle Bode plots.

9.5 Draw the magnitude and angle Bode plots of the system function

$$H(j\omega) = 9.5\frac{j\omega(1 + j\omega/50)}{[1 + j\omega/(3 \times 10^2)][1 + j\omega/(2 \times 10^3)][1 + j\omega/10^6]}$$

High-, Low-, and Midband-Frequency Limits

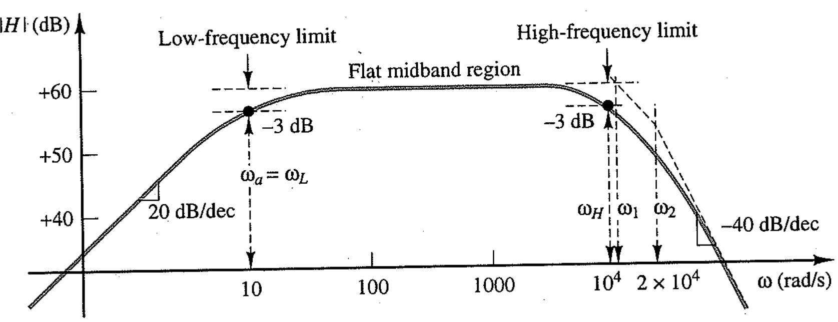
Many signal-processing applications require a circuit or system to have a flat, constant response over a specified range of frequencies called the *midband*. If the frequency components of the input signal are confined to this range, the output will replicate the form of the input and have the same spectral content. For such circuits, the locations of the specific poles and zeros of the system function are of less interest than the frequency range over which the response may be considered flat.

The flat-response region is usually the portion of the Bode plot with maximum magnitude. Its limits are therefore defined as those frequencies ω_L and ω_H at which the magnitude of the system function first falls by a factor of $1/\sqrt{2}$, or -3 dB, from the horizontal. A magnitude reduction of $1/\sqrt{2}$ corresponds to halving of the power delivered to a resistive load.

The limits of the midband region may not always coincide with individual poles. Multiple poles may contribute simultaneously to the degradation of the circuit's output amplitude. This concept is illustrated in Fig. 9.18, which depicts the magnitude Bode plot of the system function:

$$H(j\omega) = 1000\frac{j\omega/10}{(1 + j\omega/10)(1 + j\omega/10^4)[1 + j\omega/(2\times 10^4)]} \tag{9.57}$$

Equation (9.57) has a low-frequency pole at $\omega_a = 10$ rad/s and two high-frequency poles—one at $\omega_1 = 10^4$ rad/s and one at $\omega_2 = 2 \times 10^4$ rad/s.



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rad/s and
10^4 rad/s.

The low- and high-frequency limits ω_L and ω_H are used to designate the ends of the flat midband region, which has a magnitude of $|H| = 1000 \equiv +60$ dB. One might assume from the discussion of Section 9.2.1 that $\omega_1 = 10^4$ rad/s, the first pole to be encountered above the midband, represents ω_H. The system function (9.57) has another nearby pole at $\omega_2 = 2\times 10^4$ rad/s, however, which also contributes to the reduction of the Bode plot magnitude at ω_1.

The exact value of ω_H can be computed by solving for the frequency ω_H at which $|H|$ falls by $1/\sqrt{2}$ from its midband value of 1000:

$$|H|_{\omega=\omega_H} \equiv \frac{100\omega_H}{[1 + (\omega_H/10)^2]^{1/2}[1 + (\omega_H/10^4)^2]^{1/2}[1 + (\omega_H/2\times 10^4)^2]^{1/2}} = \frac{1000}{\sqrt{2}} \tag{9.58}$$

This equation can be solved for ω_H to yield

$$\omega_H \approx 0.84 \times 10^4 \text{ rad/s} \tag{9.59}$$

This frequency is lower than the breakpoint $\omega_1 = 10^4$ because the nearby pole at $\omega_2 = 2 \times 10^4$ rad/s also degrades the system response at frequencies near ω_1.

9.2.4 Superposition of Poles

For a system function like Eq. (9.57), which exhibits a clearly defined midband region, the locations of ω_H and ω_L can always be found exactly by solving an equation of the form (9.58). Such calculations, however, become tedious for system functions with many closely spaced poles. In such cases, a simplifying technique called the *superposition-of-poles* approximation provides reasonable estimates of ω_L and ω_H while eliminating much of the tedious algebra.

The superposition-of-poles approximation can be applied to any system function that can be put in the form of a midband-gain multiplied by separate low-frequency and high-frequency system functions. Such a system function will have the overall form

$$\begin{aligned} H(j\omega) &= A_o \cdot H_L \cdot H_H \\ &= A_o \underbrace{\left[\frac{(j\omega/\omega_a)}{(1+j\omega/\omega_a)}\frac{(j\omega/\omega_b)}{(1+j\omega/\omega_b)}\cdots\frac{(j\omega/\omega_m)}{(1+j\omega/\omega_m)}\right]}_{H_L} \\ &\quad \times \underbrace{\left[\frac{1}{(1+j\omega/\omega_1)(1+j\omega/\omega_2)\cdots(1+j\omega/\omega_n)}\right]}_{H_H} \end{aligned} \tag{9.60}$$

Here A_o is a constant equal to the magnitude of the system function in the flat midband region, and H_L and H_H constitute the low- and high-frequency contributions, respectively, to $H(j\omega)$. The breakpoints $\omega_a \cdots \omega_m$ of H_L jointly define the low-frequency limit of the midband. The breakpoints $\omega_1 \cdots \omega_n$ of H_H define the high-frequency limit of the midband. Note that Eq. (9.57) is of the form (9.60), with $A_o = 1000$, $\omega_a = 10$, $\omega_1 = 10^4$, and $\omega_2 = 2 \times 10^4$.

High-Frequency Limit

At the high-frequency end of the midband, the poles of H_L have no effect on the response. At $\omega = \omega_H$, for example, each of the binomials in the denominator of H_L approaches the value $j\omega_H/\omega_m$, canceling the corresponding factor $j\omega_H/\omega_m$ in the numerator of H_L, so that $|H_L| \to 1$. At frequencies near ω_H, $H(j\omega)$ therefore can be approximately expressed by

$$H(j\omega) \approx A_o H_H = \frac{A_o}{(1+j\omega/\omega_1)(1+j\omega/\omega_2)\cdots(1+j\omega/\omega_n)} \tag{9.61}$$

The denominator of Eq. (9.61) consists of a product of binomials that can be multiplied out and put in the form

$$\begin{aligned} 1 + j\omega\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\cdots+\frac{1}{\omega_n}\right) + (j\omega)^2\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_1\omega_n}+\frac{1}{\omega_2\omega_n}+\cdots+\frac{1}{\omega_j\omega_n}\right) \\ + (j\omega)^3\left(\text{terms of the form } \frac{1}{\omega_j\omega_k\omega_n}\right) + \cdots + \frac{(j\omega)^n}{\omega_1\omega_2\cdots\omega_n} \end{aligned} \tag{9.62}$$

The second term in Eq. (9.62) contains the factor $j\omega/\omega_n$ from each binomial; the third term contains all possible combinations of $\omega^2/\omega_j\omega_k$; the fourth term contains all possible combinations of order ω^3, and so on. The final term is equal to $(j\omega)^n/(\omega_1\omega_2\cdots\omega_n)$.

By definition, all of the poles ω_1 through ω_n of Eq. (9.61) are higher than the midband endpoint ω_H. Thus, at frequencies near ω_H, terms of order ω^2 or higher in Eq. (9.62) may be ignored, because these terms will be much smaller than terms of order ω. This approximation is weakest when two poles coincide exactly near ω_H, but can be shown to yield moderately good results even in such a case (see Problem 9.36). Neglecting terms of order ω^2 or higher in the denominator of equation (9.62) allows the approximate high-frequency system function (9.61) to be further approximated by

$$H(j\omega) \approx \frac{A_o}{1 + j\omega\left(\frac{1}{\omega_1} + \frac{1}{\omega_2} + \cdots + \frac{1}{\omega_n}\right)} \tag{9.63}$$

The denominator of Eq. (9.63) contains a single binomial term that causes $|H|$ to fall by -3 dB when the imaginary part of the denominator equals the real part. The high-frequency -3 dB point ω_H of the system function (9.60), which constitutes the upper limit of the midband region, will thus be given approximately by

$$\omega_H = \left(\frac{1}{\omega_1} + \frac{1}{\omega_2} + \cdots + \frac{1}{\omega_n}\right)^{-1} \tag{9.64}$$

As Eq. (9.64) suggests, ω_H can be expressed in "parallel combination" notation as $\omega_1 \| \omega_2 \cdots \| \omega_n$ and can be thought of as the "parallel" superposition of all the individual high-frequency poles ω_1 through ω_n. Equation (9.64) is known as the *superposition-of-poles* approximation at the high-frequency end of the midband.

According to (9.64), the high-frequency poles with the lowest frequency will make the most contribution to ω_H. If one pole is significantly lower in frequency, it will dominate ω_H. Similarly, poles located near each other will make nearly equal contributions to ω_H. Any poles located well above ω_H will make little contribution to the value of ω_H.

Low-Frequency Limit

A similar approach applies at the low-frequency end of the midband. Near the low-frequency end of $H(j\omega)$, the poles of the high-frequency function H_H have little effect on the response. At such frequencies, each of the binomial terms in H_H approaches unity. At frequencies near the low-frequency limit ω_L, the system function $H(j\omega)$ given by Eq. (9.60) thus can be approximately expressed by

$$H(j\omega) \approx A_o H_L = A_o \frac{(j\omega/\omega_a)}{(1 + j\omega/\omega_a)} \frac{(j\omega/\omega_b)}{(1 + j\omega/\omega_b)} \cdots \frac{(j\omega/\omega_m)}{(1 + j\omega/\omega_m)} \tag{9.65}$$

If each of the factors $j\omega/\omega_a$ through $j\omega/\omega_m$ is divided into numerator and denominator, Eq. (9.65) becomes

$$H(j\omega) \approx A_o H_L = A_o \frac{1}{(\omega_a/j\omega + 1)(\omega_b/j\omega + 1) \cdots (\omega_m/j\omega + 1)} \tag{9.66}$$

The denominator of Eq. (9.66) can be expressed in polynomial form as

$$\begin{aligned} &1 + \frac{1}{j\omega}(\omega_a + \omega_b + \cdots + \omega_m) + \frac{1}{(j\omega)^2}(\omega_a\omega_b + \omega_a\omega_m + \omega_b\omega_m + \cdots + \omega_j\omega_m) + \\ &\cdots + \left[\text{terms of the form} \frac{1}{(j\omega)^3}(\omega_j\omega_k\omega_m)\right] + \cdots + \frac{1}{(j\omega_L)^m}(\omega_a\omega_b \cdots \omega_m) \end{aligned} \tag{9.67}$$

By definition, all the poles $\omega_a \cdots \omega_m$ are lower in frequency than the actual low-frequency midband endpoint ω_L. Hence, at frequencies near $\omega = \omega_L$, the terms of order $1/\omega^2$ or higher may be ignored. These terms are presumed to be much smaller than terms of order $1/\omega$. This approximation is weakest when two poles coincide exactly near ω_L. It can be shown, however, that the approximation yields good results even in such a case (see Problem 9.36).

Neglecting terms of order $(1/\omega)^2$ or higher allows the approximate low-frequency system function (9.65) to be further approximated by

$$H(j\omega) \approx A_o \frac{1}{1 + (1/j\omega)(\omega_a + \omega_b + \cdots + \omega_m)}$$

Multiplying numerator and denominator by $j\omega$ and dividing both by $(\omega_a + \omega_b + \cdots + \omega_m)$ results in

$$H(j\omega) = \frac{j\omega/(\omega_a + \omega_b + \cdots + \omega_m)}{1 + [j\omega/(\omega_a + \omega_b + \cdots + \omega_m)]} \tag{9.68}$$

The denominator of this expression contains a single complex binomial that describes the lower −3-dB endpoint of the system function (9.60). According to Eq. (9.68), the value of this low-frequency limit will be given approximately by

$$\omega_L \approx \omega_a + \omega_b + \cdots + \omega_m \tag{9.69}$$

As this expression suggests, the low-frequency −3-dB limit of the midband region may be expressed as an additive "series" superposition of all the low-frequency poles $(\omega_a \cdots \omega_m)$. As indicated by Eq. (9.69), the low-frequency poles with the highest value will make the most contribution to ω_L. If one pole is significantly higher in frequency, it will dominate. Poles located near each other will contribute in nearly equal amounts to ω_L. Similarly, any poles located well below ω_L will make little contribution to the value of ω_L.

Summary of Method

In summary, when a system function has a clearly defined midband region, the superposition-of-poles approximation may be applied by classifying all poles as either high- or low-frequency types. The upper −3-dB point ω_H of the flat midband region can be estimated by a parallel superposition of poles:

$$\omega_H \approx \frac{1}{1/\omega_1 + 1/\omega_2 + \cdots + 1/\omega_n} \equiv \omega_1 \| \omega_2 \cdots \| \omega_n \tag{9.70}$$

The lower −3-dB point ω_L of the flat midband region can be estimated by a series superposition of poles:

$$\omega_L \approx \omega_a + \omega_b + \cdots + \omega_m \tag{9.71}$$

If multiple poles exist at either end of the midband, the superposition-of-poles approximation will always slightly *underestimate* the actual value of ω_H and slightly *overestimate* the actual value of ω_L.

EXAMPLE 9.3 Use the superposition-of-poles approximation to estimate the upper −3-dB endpoint ω_H of the system function:

$$H(j\omega) = 1000 \frac{j\omega/10}{(1 + j\omega/10)(1 + j\omega/10^4)[1 + j\omega/(2 \times 10^4)]} \tag{9.72}$$

Compare the result to the true value (9.59) obtained from Eq. (9.58). (This system function contains only one low-frequency pole, hence the superposition-of-poles approximation is not needed to find ω_L.)