CrossMark
click for updates

On the linkage between the $k^{-5/3}$ spectral and $k^{-7/3}$ cospectral scaling in high-Reynolds number turbulent boundary layers

Dan Li[1] and Gabriel G. Katul[2,3]
[1]*Department of Earth and Environment, Boston University, Boston, Massachusetts 02215, USA*
[2]*Nicholas School of the Environment, Duke University, Durham, North Carolina 27708, USA*
[3]*Department of Civil and Environmental Engineering, Duke University, Durham, North Carolina 27708, USA*

Connections between the "−5/3" spectral and "−7/3" cospectral scaling exponents characterizing the inertial subranges of the wall-normal energy spectrum and the turbulent momentum flux cospectrum are explored in the equilibrium layer of high-Reynolds number turbulent boundary layers. Previous laboratory experiments and field measurements featured here in the atmospheric boundary layer show that the "−7/3" scaling in the momentum flux cospectrum $F_{uw}(k)$ commences at lower wavenumbers (around $kz = 3$) than the "−5/3" scaling in the wall-normal energy spectrum $E_{ww}(k)$ (around $kz = 6$), where k is the streamwise wavenumber and z is the distance from the surface. A satisfactory explanation as to why $F_{uw}(k)$ attains its "−7/3" inertial subrange scaling earlier than $E_{ww}(k)$ in wavenumber space remains elusive. A cospectral budget (CSB) model subject to several simplifications and closure schemes offers one viewpoint. In its simplest form, the CSB model assumes a balance at all k between the production term and a Rotta-like pressure decorrelation term with a prescribed wavenumber-dependent relaxation time scale. It predicts the "−7/3" scaling for $F_{uw}(k)$ from the "−5/3" scaling in $E_{ww}(k)$, thereby recovering earlier results derived from dimensional considerations. A finite flux transfer term was previously proposed to explain anomalous deviations from the "−7/3" cospectral scaling in the inertial subrange using a simplified spectral diffusion closure. However, this explanation is not compatible with an earlier commencement of the "−7/3" scaling in $F_{uw}(k)$. An alternative explanation that does not require a finite flux transfer is explored here. By linking the relaxation time scale in the slow-component of the Rotta model to the turbulent kinetic energy (TKE) spectrum, the earlier onset of the "−7/3" scaling in $F_{uw}(k)$ is recovered without attainment of a "−5/3" scaling in $E_{ww}(k)$. The early onset of the "−7/3" scaling at smaller k is related to a slower than $k^{-5/3}$ decay in the TKE spectrum at the crossover from production to inertial scales. *Published by AIP Publishing.*
[http://dx.doi.org/10.1063/1.4986068]

I. INTRODUCTION

The spectral characteristics of high-Reynolds number turbulent boundary layers have been extensively studied in the inertial subrange, where the well-known Kolmogorov scaling laws (hereafter referred to as K41) appear to hold.[1] Compared to the accepted "−5/3" scaling describing the decay of energy spectra of all velocity components with increasing wavenumber k within the inertial subrange, the scaling laws of the cospectra as well as the range of eddy sizes they describe are under-explored and perhaps more controversial. In the presence of mean velocity gradients acting on all k within the inertial subrange, dimensional considerations alone[2,3] predicted a "−7/3" scaling for the cross-stream momentum cospectrum. These cospectral predictions received some verification from laboratory and field experiments.[4,5] Since then, the "−7/3" scaling has been widely used to adjust measured turbulent fluxes due to high-frequency losses resulting from instrument separation and path-length averaging. In fact, these corrections are now part of the standard post-processing protocol in eddy-covariance turbulent flux monitoring of momentum, heat, water vapor, carbon dioxide, ozone, methane, isoprene, volatile organic

compounds (VOC) including biogenic VOCs, and many other gaseous compounds.[6] However, the "−7/3" scaling for scalars is not universally supported with some laboratory experiments reporting a "−2" scaling instead for the cross-stream heat flux cospectrum.[7]

Based on a cospectral budget (CSB) model, recent studies showed that the scaling law of the cross-stream scalar flux cospectrum in isotropic turbulence depends on contributions from the flux transfer and pressure-decorrelation terms,[8] and is Reynolds number-dependent.[9] A similar CSB analysis has been extended to the momentum flux,[10] which is also the focus of our study here, in the atmospheric surface layer[11–15] and in the canopy sublayer.[16] The outcome of these aforementioned CSB analyses raises the possibility of deviations from "−7/3" scaling for momentum and heat flux cospectra in the inertial subrange where the energy spectra of all velocity components maintain their accepted "−5/3" scaling.[14–16] Based on these studies,[14–16] a "−5/3" scaling in the energy spectrum may not be a *sufficient* condition for the onset of a "−7/3" cospectral scaling. Data reported and analyzed here also support this statement. It is then natural to raise the question as to whether the "−5/3" scaling in the energy spectrum predicted by K41

is a *necessary* condition for the occurrence of a "−7/3" scaling in the flux cospectrum, which has not been explored. This is also motivated by the fact that very high-Reynolds number wind-tunnel experiments, routinely cited in turbulence reference books,[17] exhibited a "−7/3" scaling in the momentum flux cospectrum at wavenumbers smaller than their "−5/3" counterpart in the wall-normal energy spectrum.[5]

To answer this question requires a re-examination of the relation between the "−7/3" and "−5/3" scaling laws for the cospectrum and energy spectrum, respectively, especially at the crossover from production to inertial scales and throughout the inertial subrange. The focus here is on the vertical momentum flux cospectrum $F_{uw}(k)$ and its connection with the wall-normal energy spectrum $E_{ww}(k)$ in an idealized atmospheric surface layer (ASL) flow that is stationary, horizontally homogeneous, lacking any subsidence or large-scale mean pressure gradients, and characterized by very high-Reynolds numbers so that the scale separation between production and dissipation ranges is sufficiently large. The streamwise energy spectrum $E_{uu}(k)$ is not examined as it does not generate vertical momentum flux in such an idealized ASL flow. The study makes use of multi-level sonic anemometer measurements conducted above a uniform lake, where the nearly constant momentum flux assumption with distance from the surface z was checked prior to the analysis. According to the time-averaged longitudinal momentum balance, a constant turbulent momentum flux in the vertical direction signifies that the flow is stationary, horizontally homogeneous, and lacking any subsidence or large-scale mean pressure gradients.[18] The CSB model subject to the aforementioned simplifications for an idealized ASL is then employed to link the spectral and cospectral exponents. In essence, the CSB model is used to diagnose the potential causes of an earlier onset (i.e., at smaller wavenumbers) of the "−7/3" scaling in the flux cospectrum than the "−5/3" scaling in the energy spectrum. Because the path length of a sonic anemometer used to measure velocity fluctuations is comparable to or larger than the Taylor microscale, no attempts have been made to explore the spectral and cospectral properties near the crossover from inertial to viscous subranges.

II. EXPERIMENT

The three velocity components and air temperature were measured using sonic anemometry above Lake Leman (also

known as Lake Geneva) situated between Switzerland and France on a 10 m tall tower and were described elsewhere.[19–24] Briefly, time series of the three velocity components and virtual temperature were collected during the late summer and autumn of 2006 but only runs with mean wind direction originating from the south and southwest were employed to minimize any tower interference and ensure extensive fetch (on the order of 10 km). The measurements were conducted at 4 heights (1.65, 2.30, 2.95, and 3.60 m) above the water surface, and only runs where the measured turbulent momentum flux did not vary by more than 10% with height were used here. Again, it is assumed that the constant turbulent momentum flux in the vertical direction signifies that the flow is stationary, horizontally homogeneous, and lacks any subsidence or large-scale mean pressure gradients. These assumptions were also employed in the development of the CSB model, thus necessitating the filtering of runs that do not exhibit approximately constant momentum flux with z. During the experiment, the water surface principally responded to the local wind conditions and the wave height, characterized by a median of 0.03 m, rarely exceeded 0.2 m.[19] As a result, the waves were much smaller than the measurement height or the scales of interest in our study. The sampling frequency and averaging intervals for each run were 20 Hz and 30 min, respectively. Taylor's frozen turbulence hypothesis[25] was used to convert time series to one-dimensional, streamwise wavenumbers k using the mean velocity for each run as is common in such field experiments.

The compensated vertical (or wall-normal) velocity spectrum E_{ww} and momentum flux cospectrum F_{uw} as a function of k are shown in Fig. 1. The reported spectra and cospectrum are ensemble averaged over 97 runs. These runs are selected following the same quality control measures detailed in previous studies[20,23] and are not repeated here. Beyond these quality controls, only data under near-neutral atmospheric stability conditions are considered as identified by the absolute value of the atmospheric stability parameter being smaller than 0.05. The atmospheric stability parameter measures the ratio of the buoyant generation (or destruction) to the mechanical generation of TKE at the measurement height z. The data here illustrate that E_{ww} attain a "−5/3" scaling in the wavenumber range of $6 < kz < 12$ (indicated by the vertical lines). However, F_{uw} attains a "−7/3" scaling at smaller wavenumbers (around $kz = 3$). Note that the exact values of
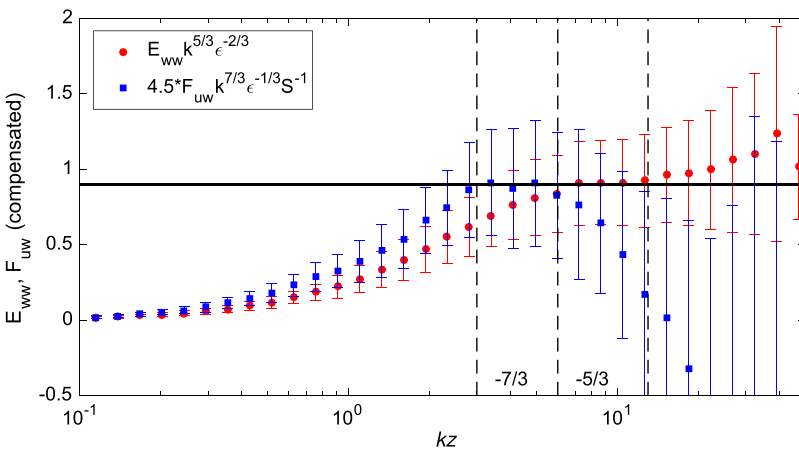

FIG. 1. The compensated vertical (or wall-normal) velocity spectra E_{ww} and the vertical momentum flux cospectra F_{uw} in the range of $0.01 < kz < 50$. The results are averaged over 97 segments of 30-min turbulence data sampled at 20 Hz above a uniform lake. Bars denote the standard deviation in each bin. The compensated flux cospectra F_{uw} are multiplied by 4.5 so that the ranges of E_{ww} and F_{uw} are similar. The horizontal line indicates the value of 0.9, which is slightly larger than the Kolmogorov constant reported in the literature[4,5,26] when k is interpreted as a one-dimensional streamwise wavenumber. The vertical lines indicate $kz = 3$, $kz = 6$, and $kz = 12$, respectively.
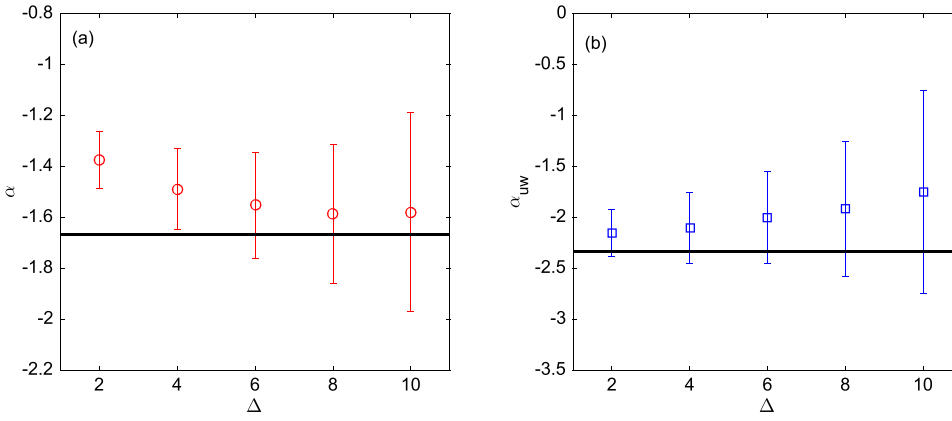
FIG. 2. The scaling exponents (a) α and (b) α_{uw} of the wall-normal velocity spectra E_{ww} and the vertical momentum flux cospectra F_{uw} determined by fitting $E_{ww}(k) = C_{ww}k^{-\alpha}$ and $F_{uw}(k) = C_{uw}k^{-\alpha_{uw}}$ to k as a function of the lower-bound Δ, where the regression range spans $\Delta < kz < 15$. Bars denote the standard deviation across 97 runs.

kz at which the "−5/3" and "−7/3" scaling laws appear depend on how one interprets the uncertainties as indicated by the error bars. Here, the mean values in each wavenumber bin class are used as the primary indicator. This finding suggests that there is a small wavenumber range ($3 < kz < 6$) in which the cospectrum exhibits a "−7/3" scaling but the energy spectrum does not exhibit a "−5/3" scaling. The main objective of the work here is to explore why the "−7/3" scaling occurs at smaller wavenumbers in F_{uw} than the "−5/3" scaling in E_{ww}.

Besides the main objective here, it is worth noting that the lake data support previous findings[8,14–16,27] that a vertical velocity spectrum exhibiting a "−5/3" scaling is not sufficient for the cospectrum to display a "−7/3" scaling. Specifically, in the wavenumber range where the "−5/3" scaling exists in E_{ww} (i.e., $6 < kz < 12$), the "−7/3" scaling is not apparent in F_{uw}.

The spectral (and cospectral) scaling component for each individual run is separately computed based on fitting the spectrum (and cospectrum) with an expression of the form $E_{ww}(k) = C_{ww}k^{-\alpha}$ (and $F_{uw}(k) = C_{uw}k^{-\alpha_{uw}}$) in the wavenumber range of $\Delta < kz < 15$, where Δ varies. The fitting is carried out in logarithmic space using the polyfit function in Matlab (Mathworks, Release 2016a). The lower bound for the commencement of the inertial subrange ($kz = 2$–3) is motivated by Fig. 1. It is also consistent with other studies identifying the inertial subrange based on isotropy considerations using component-wise velocity structure functions.[28] The upper limit is based on the fact that the sonic anemometer path length is 0.1 m, which corresponds to $kz \approx 15$ when accounting for aliasing effects and using the measurement height of $z \approx 3$ m. As a result, the choice of $kz \approx 15$ may be sufficient to minimize the effect of instrument path averaging. Figure 2 shows how the scaling components (α for E_{ww} and α_{uw} for F_{uw}) change with increasing Δ. As Δ increases, the spectral scaling approaches "−5/3" consistent with expectations from K41 theory. The "−5/3" scaling is not reached until $\Delta = 6$. On the contrary, the cospectral scaling appears to deviate from "−7/3" with increasing Δ, at least the mean values across all the runs. Note that the uncertainties, represented by the error bars, increase as Δ increases. This increase is expected because the number of data points as well as the wavenumber separation used to constrain the fitting in logarithmic space is reduced. Again, the fitting is performed in the

range of $\Delta < kz < 15$. Despite these uncertainties, the trends are evident.

This phenomenon of the momentum flux cospectrum exhibiting a "−7/3" scaling prior to the commencement of the "−5/3" scaling in the wall-normal energy spectrum is not peculiar to the field experiment here. It has been observed in benchmark high-Reynolds number laboratory experiments before.[5] To date, however, no satisfactory explanation has been offered, which is the main motivation of the work here.

III. THEORY

The analysis is restricted to the cospectrum in a high-Reynolds number, neutrally stratified ASL flow assumed to be stationary, horizontally homogenous, and lacking any subsidence or large-scale mean pressure gradients. The focus here is on the measured one-dimensional cospectrum that satisfies the normalizing property $\int_0^\infty F_{uw}(k)dk = \overline{u'w'}$, where the overbar indicates Reynolds (or time) averaging. Within the confines of these assumptions and the normalizing property, the CSB model linking the spectral and cospectral scaling exponents is now briefly reviewed. At a given wavenumber k, the CSB model for $F_{uw}(k)$ is given by[8,11,27]

$$\frac{\partial F_{uw}}{\partial t} = 0 = P_{uw} + T_{uw} + \pi_{uw} - 2\nu k^2 F_{uw}. \qquad (1)$$

The term on the left hand-side represents the temporal change of F_{uw}. The terms on the right hand-side represent (in order) shear or mechanical production (P_{uw}), flux transfer (T_{uw}), pressure decorrelation (π_{uw}), and viscous destruction ($2\nu k^2 F_{uw}$, where ν is the kinematic viscosity) that is expected to be small for high-Reynolds number flows. With this assumption, the cospectral budget for the momentum flux simplifies to

$$P_{uw} + T_{uw} + \pi_{uw} = 0. \qquad (2)$$

By performing Fourier transforms of the shear production term in physical space and invoking a Rotta model[17] for parameterizing the pressure decorrelation term (π_{uw}) and a spectral gradient diffusion model[8] for parameterizing the flux transfer term (T_{uw}), the following expressions can be derived:

$$P_{uw}(k) = -E_{ww}(k)\,S, \qquad (3)$$

$$\pi_{uw}(k) = -A_U \frac{F_{uw}(k)}{\tau(k)} - \frac{3}{5}P_{uw}, \quad (4)$$

$$T_{uw}(k) = -A_{UU} \frac{\partial}{\partial k} \left[\frac{kF_{uw}(k)}{\tau(k)} \right], \quad (5)$$

where S is the mean streamwise velocity gradient along the vertical direction, A_{UU} is a parameter in the gradient diffusion model for the flux transfer term, A_U is the Rotta parameter associated with the slow-component of π_{uw},[29] whereas the $(3/5)P_{uw}$ term is associated with the fast-component (wall blocking is ignored at z sufficiently far from the boundary for reasons discussed elsewhere[30]), and $\tau(k) = \epsilon^{-1/3}k^{-2/3}$ is a wavenumber-dependent relaxation time scale derived from K41 dimensional considerations[31] where ϵ is the mean TKE dissipation rate. In statistical mechanics, the relaxation time scale measures the time required for a perturbed system to go back to an equilibrium state. In turbulence, equilibrium may be referenced to the state where turbulent fluctuations are dissipated by the action of ν. Hence, with a constant ϵ across scales or k (i.e., a conservative cascade), $\tau(k) = \epsilon^{-1/3}k^{-2/3}$ may be viewed as a measure of the mean duration of an eddy of size k^{-1} to be dissipated (regardless of its turbulent energy content) as discussed elsewhere.[32] Alternative definitions that accommodate the turbulent energy content at k are discussed later on. With this definition of $\tau(k)$, the relative importance of the slow-component of π_{uw} and the viscous term $2\nu k^2 F_{uw}(k)$ is given as

$$\frac{2\nu k^2 F_{uw}(k)}{A_U F_{uw}(k)/\tau(k)} = \frac{2}{A_U}\left(\frac{\nu^3 k^4}{\epsilon}\right)^{1/3} \sim (k\eta)^{4/3}, \quad (6)$$

where $\eta = (\nu^3/\epsilon)^{1/4}$ is the Kolmogorov microscale, which is ~ 0.1–1 mm in ASL flows.[28] For $k\eta \ll 1$ (as is the case here and in K41), the decorrelation due to viscous effects can be ignored relative to the linear slow part of the Rotta term. However, as $k\eta \gtrsim 1$, the two terms become comparable in magnitude, but at such fine scales, $|F_{uw}(k)|$ is already sufficiently small so that ignoring its contribution to $\overline{u'w'}$ is justifiable.

Inserting the aforementioned closures in the CSB model yields

$$\frac{\partial F_{uw}(k)}{\partial k} + D_1 \frac{F_{uw}(k)}{k} = D_2 E_{ww}(k) k^{-\frac{5}{3}}, \quad (7)$$

with

$$D_1 = \frac{A_U}{A_{UU}} + \frac{5}{3}; \quad D_2 = -\frac{2}{5}\frac{1}{A_{UU}}\frac{S}{\epsilon^{1/3}}. \quad (8)$$

The general solution to this ordinary differential equation (ODE) is

$$F_{uw}(k) = k^{-D_1}\left(D_2 \int_0^k p^{-\frac{5}{3}+D_1} E_{ww}(p)\,dp + E_1\right), \quad (9)$$

where E_1 is an integration constant arising from the consideration of flux transfer among scales and can be determined from the normalizing property of the cospectrum. In the case where $S \approx 0$, $F_{uw}(k) = E_1 k^{-D_1}$ and represents the homogeneous solution of the aforementioned ODE. Physically, this homogeneous solution represents the balance between the flux transfer and pressure decorrelation terms when a finite $F_{uw}(k_o)$ is introduced at some reference low-wavenumber k_o. While the general solution is valid for any

$E_{ww}(k)$ shape, progress on the study objective here benefits from considering the specific case when $E_{ww}(k) = C_{ww}k^{-\alpha}$, where α is the spectral scaling and C_{ww} is the normalization factor. For this $E_{ww}(k)$ shape, the general solution reduces to

$$F_{uw}(k) = \frac{D_2}{-\alpha - 2/3 + D_1}C_{ww}k^{-\alpha-2/3} + E_1 k^{-D_1}. \quad (10)$$

As before, the $E_1 k^{-D_1}$ term is the homogeneous solution of the ODE reflecting the balance between flux transfer and pressure decorrelation terms. The $D_2(-\alpha - 2/3 + D_1)^{-1}C_{ww}k^{-\alpha-2/3}$ term modifies this homogeneous solution because of the shear production at all k due to finite S.

In the inertial subrange where $\alpha = 5/3$ and $C_{ww} = C_o\epsilon^{2/3}$ (C_o is the Kolmogorov constant for the vertical velocity energy spectrum), the cospectrum is given by

$$F_{uw}(k) = -D_3 S\epsilon^{1/3}k^{-7/3} + E_1 k^{-\left(\frac{A_U}{A_{UU}}+\frac{5}{3}\right)}, \quad (11)$$

where

$$D_3 = \frac{2}{5}\frac{\frac{C_o}{A_{UU}}}{\frac{A_U}{A_{UU}} - \frac{2}{3}}. \quad (12)$$

The above analysis makes clear that anomalous scaling (or deviations from the "−7/3" scaling) in F_{uw} is expected when $A_U/A_{UU} \neq 2/3$. This finding again suggests that a "−5/3" scaling in E_{ww} is not sufficient for the cospectrum to display a "−7/3" scaling[14–16] as shown in Fig. 1 in the wavenumber range of $6 < kz < 12$. The appearance of A_U/A_{UU} is directly related to the flux transfer across scales. Physically, this ratio A_U/A_{UU} determines whether the decorrelation (or the degeneration of flux) is caused by pressure variations or transfer towards other scales. The suggested value of A_U is about 1.8,[17] which was shown to recover the von-Kámán constant when using the accepted value of C_o.[10] However, the value of A_{UU} is ill-constrained theoretically and experimentally. In the following, the two situations where $A_U/A_{UU} > 2/3$ and $A_U/A_{UU} < 2/3$ are discussed.

When $A_U/A_{UU} > 2/3$, the anomalous scaling component decays faster than "−7/3" with increasing k. Under such conditions, $D_3 > 0$ and the first part of the cospectral solution is down-gradient, implying that the main terms in balance are the shear production and pressure decorrelation. When $A_{UU} \to 0$ ($A_U/A_{UU} \to \infty$), the flux transfer term can be completely ignored and F_{uw} follows the "−7/3" scaling, thereby recovering the classical results derived from dimensional analysis.[2]

When $A_U/A_{UU} < 2/3$, the anomalous scaling term drops slower than "−7/3." D_3 is negative, so the first part of the cospectrum is, in fact, up-gradient. Despite its counter-gradient contribution, the shear production term here may not play a significant role with increasing k because the anomalous scaling term decays slower than "−7/3," and the dominant terms in the cospectral budget are expected to be the flux transfer and pressure decorrelation terms (i.e., the homogeneous solution of the CSB).

The above two situations correspond to the case where $\alpha = 5/3$. On the other hand, when $\alpha \neq 5/3$, the cospectrum can maintain a "−7/3" scaling only when

A_U/A_{UU} = 2/3 and $D_2(-\alpha - 2/3 + D_1)^{-1}C_{ww} \ll E_1$ [see Eq. (10)].[16] This condition operates regardless of the uncertainties in A_U and A_{UU}. This condition is rather stringent as it involves a delicate balance between the flux transfer and pressure decorrelation terms, which gives rise to A_U/A_{UU} = 2/3. It also requires a negligible shear production contribution to the CSB or, at minimum, $D_2(-\alpha - 2/3 + D_1)^{-1}C_{ww} \ll E_1$, which are unlikely in a near-neutral ASL. These are some of the reasons why an alternative explanation is sought.

IV. AN ALTERNATIVE EXPLANATION

The conditions on the flux transfer term provided by the aforementioned CSB model appear overly stringent for the cospectrum to exhibit a "–7/3" scaling when the wall-normal velocity spectrum does not exhibit a "–5/3" scaling. Here, an alternative explanation is explored, which pertains to the choice of the relaxation time scale in the linear part of the Rotta model used to close the pressure-decorrelation term. Originally, this term was closed by assuming a relaxation time scale $\tau = \epsilon^{-1/3}k^{-2/3}$ based on K41 arguments. There are two drawbacks to this representation of $\tau(k)$. The first drawback is that K41 scaling is *a priori* encoded in $\tau(k)$, so it is not straightforward to demonstrate why a "–7/3" cospectral scaling occurs without a "–5/3" scaling in the energy spectrum. The second is that $\tau(k)$ ignores the actual kinetic energy content in eddies, which may be a significant consideration near the crossover from production to inertial scales.

Hence, few revisions are now made to the CSB model so as to explore an alternative explanation to the early onset of the "–7/3" scaling in spectral space. First, the relaxation time scale $\tau(k)$ is replaced by a form that depends on the TKE spectral shape. Instead of a constant ϵ dictating the entire turnover time at all k for $kz > 2$, a turnover velocity v_k based on the size (or k) and the TKE content of eddies is employed.[31] It is assumed that $\tau(k)$ reflects a turnover time scale formed by v_k and k given by $\tau(k) = 1/(kv_k)$. The TKE of eddies of size k is used to estimate v_k following $v_k^2 = kE_{TKE}(k)$. This choice leads to $\tau(k) = 1/(kv_k)$ = $\left[k^3 E_{TKE}(k)\right]^{-1/2}$. Clearly, this representation recovers $\tau(k) = \epsilon^{-1/3}k^{-2/3}$ when K41 scaling is applied to $E_{TKE}(k)$. Hence, in the inertial subrange, these two formulations for $\tau(k)$ converge, but at the crossover from production to inertial scales, they may yield different results depending on how $E_{TKE}(k)$ scales with k.

Other parameterizations for $\tau(k)$ are also tested such as a non-local version of relaxation time scale[31,33–36] given by

$$\tau_{NL}(k) = \left[\int_{k_p}^{k} p^2 E_{TKE}(p)\,dp\right]^{-1/2}, \quad (13)$$

where p is a dummy integration variable and k_p is interpreted as a finite low wavenumber limit to the production. In this study, $k_p = 0$ is assumed (i.e., far from the crossover from production to inertial). The $\tau_{NL}(k)$ assumes that turbulent eddies at wavenumber k are strained by all larger-scale eddies up to k_p. This formulation for $\tau_{NL}(k)$ does not preclude non-local

interactions among scales and thus is labeled as non-local (NL).

Second, the flux transfer term is entirely dropped as a limiting case to exploring the consequences of the aforementioned revised $\tau(k)$ on the early onset of the "–7/3" scaling. This assumption is made for two reasons: to simplify the analysis and to distinguish the work here from previous studies that used the CSB model with finite flux transfer. As mentioned earlier, previous studies attributed the entire anomalous cospectral scaling in the inertial subrange to the finite flux transfer term. By ignoring the flux transfer term entirely, we can directly answer whether the early onset of the "–7/3" scaling in the cospectrum requires a finite flux transfer term or not.

Starting from the balance between the shear production and the pressure decorrelation terms ($P_{uw} + \pi_{uw} = 0$) and assuming $\tau(k) = \left[k^3 E_{TKE}(k)\right]^{-1/2}$ result in

$$F_{uw}(k) = -\frac{2}{5}\frac{1}{A_U}SE_{ww}(k)[E_{TKE}(k)]^{-1/2}k^{-3/2}. \quad (14)$$

Note that at the crossover from production to inertial scales, the scaling of $E_{ww}(k)$ might be different from that of $E_{TKE}(k)$ and both scaling laws might be different from "–5/3." To derive a general relation between the exponents, we denote $E_{ww}(k) = C_{ww}k^{-\alpha}$ and $E_{TKE}(k) = C_{TKE}k^{-\alpha_{TKE}}$, which yields

$$F_{uw}(k) = -\frac{2}{5}\frac{C_{ww}}{C_{TKE}^{1/2}A_U}Sk^{-(\alpha+\frac{3}{2}-\frac{1}{2}\alpha_{TKE})}. \quad (15)$$

To recover a "–7/3" scaling, $\alpha_{TKE}/2 - \alpha - 3/2 = -7/3$, or $\alpha_{TKE} = 2\alpha - 5/3$. It is clear that this condition is satisfied when $\alpha = \alpha_{TKE} = 5/3$, recovering the results of the original CSB model when $A_{UU} = 0$.

The condition $\alpha_{TKE} = 2\alpha - 5/3$ is much less stringent than $\alpha = \alpha_{TKE} = 5/3$, implying that a "–7/3" cospectral scaling may not require a precise "–5/3" scaling in the wall-normal energy spectrum. When $\alpha < 5/3$, a "–7/3" scaling can still be maintained in the cospectrum provided $\alpha_{TKE} = 2\alpha - 5/3 < 5/3$. Figure 3(a) shows the ensemble averaged vertical velocity (E_{ww}) and TKE (E_{TKE}) spectra in the lake data set, with the "–5/3" scaling shown as a reference. Also shown is the "–1" scaling anticipated at low wavenumbers ($kz < 1$) for near-neutral conditions. In the range of $3 < kz < 6$ where the averaged cospectrum exhibits a "–7/3" scaling, neither E_{ww} nor E_{TKE} exhibit a clear K41 "–5/3" scaling. However, deviations of both scaling exponents from "–5/3" appear to be consistent with the requirement $\alpha_{TKE} = 2\alpha - 5/3$ as both α_{TKE} and α are smaller than 5/3.

To further expand upon this result, the original model ($\tau(k) = \epsilon^{-1/3}k^{-2/3}$), the revised model ($\tau(k) = \left[k^3 E_{TKE}(k)\right]^{-1/2}$), and the non-local model (τ_{NL}) are compared in Fig. 3(b). As can be seen, the τ in the revised model and the non-local model decay faster in the range of $1 < kz < 6$, which compensates for the slower decay in E_{ww}, thereby potentially maintaining a near "–7/3" scaling in F_{uw}. Note that the non-local relaxation time scale has the same k dependence as the parameterization $\tau = \left[k^3 E_{TKE}(k)\right]^{-1/2}$ when E_{TKE} follows the "–1" scaling, as shown in Fig. 3(b) for $kz < 1$.
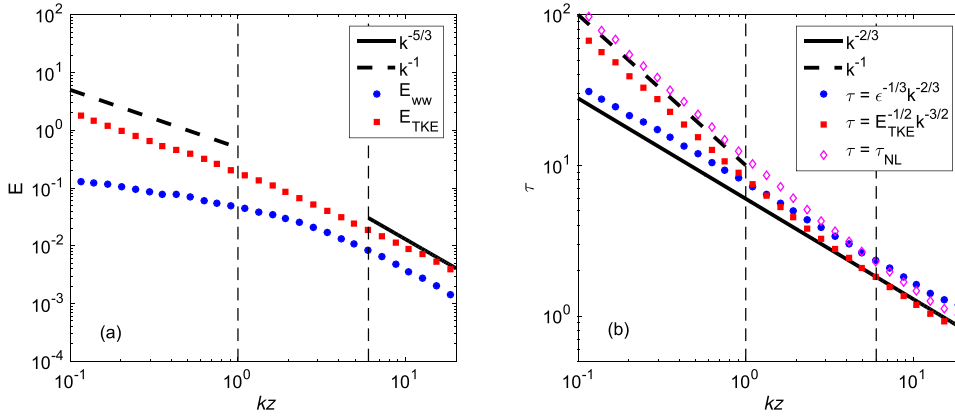
FIG. 3. (a) The measured vertical (or wall-normal) velocity (E_{ww}) and turbulent kinetic energy (E_{TKE}) spectra and (b) the modeled relaxation time scales (τ) as a function of the normalized wavenumber kz in the range of $0.1 < kz < 20$. Results are averaged over 97 segments of 30-min runs from the lake data. The $k^{-5/3}$ and k^{-1} scaling are shown for reference in (a); the $k^{-2/3}$ and k^{-1} scaling are shown for reference in (b). The vertical lines indicate $kz = 1$ and $kz = 6$, respectively.

Before discussing the modeled cospectra in Fig. 4, it is to be noted that the value of $A_U = 1.8$ is the same in all three models so that the cospectra from the three models have slightly different magnitudes, although they all agree reasonably with the measured cospectrum. The exact value of A_U does not affect the scaling laws. It is clear from Fig. 4, especially Fig. 4(b) where the compensated cospectra are shown, that the original model does not capture the "−7/3" scaling in the range of $3 < kz < 6$. The non-local model, which integrates the TKE content at low frequencies, shows the best performance in capturing the "−7/3" scaling in the range of $3 < kz < 6$. The revised model seems to fall in between, which is expected since it only considers the local TKE content. The different modeled cospectra in the range of $3 < kz < 6$ demonstrate the influence of the TKE spectrum on the scaling of momentum flux cospectrum via the τ in the slow part of the pressure redistribution term. The non-local model reasonably captures the measurements in the transition from production to inertial scales, providing a plausible justification for the early appearance of the "−7/3" cospectral scaling without requiring a finite flux transfer term.

The modeled cospectra all show a "−7/3" scaling in the range of $6 < kz < 12$. Again, the measurements do not display a "−7/3" scaling as already shown earlier in Fig. 1. This departure occurs because all three models have neglected the flux transfer term. When $kz < 1$, the ensemble-averaged TKE spectrum follows a "−1" scaling widely reported in many laboratory and several field experiments[35,37–46] as well as large eddy simulations,[47] as shown in Fig. 3(a). In contrast, the

averaged wall-normal velocity spectrum appears to be flat with no apparent scaling law.[23] Since $\alpha_{TKE} = 1$ and $\alpha = 0$ do not satisfy $\alpha_{TKE} = 2\alpha - 5/3$, the cospectrum in this range does not exhibit a "−7/3" scaling (not shown) as expected. Neither the original model nor the revised model captures the scaling of the measured cospectrum at very low wavenumber hinting that other processes must be at play, including the flux transfer term. In fact, Direct Numerical Simulation (DNS) runs[8] (see their Fig. 3) demonstrate that at very low wavenumbers, the CSB model is mainly governed by a balance between production and flux transfer terms, which offers a plausibility argument as to why the models neglecting the flux transfer terms fail. Interestingly, in the vicinity of $kz = 1$, the DNS results also suggest that the flux transfer term gradually changes sign from negative to positive with increasing k. Hence, in the wavenumber range where the "−7/3" scaling is observed in the lake data ($3 < kz < 6$), the flux transfer term may still be smaller compared to the production term, justifying its removal in the alternative explanation.

To sum up, it is conjectured that the earlier appearance of a "−7/3" cospectral scaling in the wavenumber range where the "−5/3" scaling is not present in the wall-normal velocity spectrum is related to the lack of attainment of the "−5/3" scaling in the TKE spectrum. With the TKE spectrum decaying slower than $k^{-5/3}$, the eddy turnover time scale decays faster than its K41 expectation of $k^{-2/3}$ with increasing k. As a result, the decay of cospectrum can be maintained at near $k^{-7/3}$ despite a slower than $k^{-5/3}$ decay rate in the wall-normal energy spectrum.
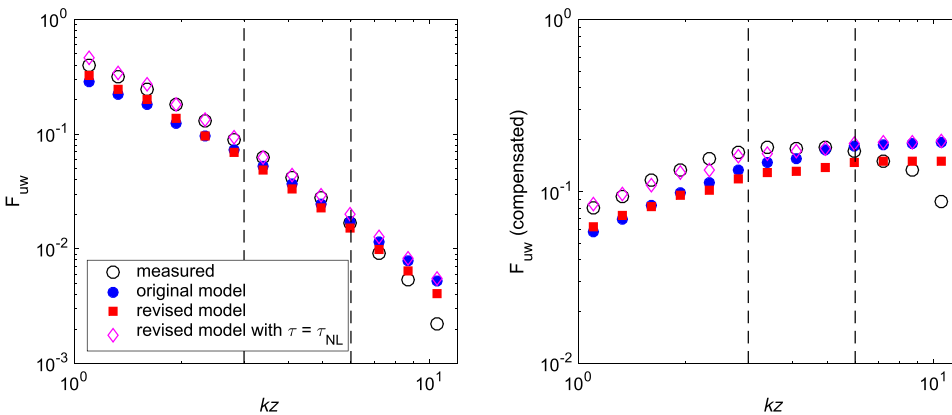

FIG. 4. A comparison between measured and modeled cospectra $F_{uw}(k)$ as a function of the normalized wavenumber kz in the range of $1 < kz < 12$. The cospectral budget model considers three relaxation time scales: $\tau = \epsilon^{-1/3}k^{-2/3}$ (labeled as original), $\tau = [k^3 E_{TKE}(k)]^{-1/2}$ (labeled as revised), and $\tau = [\int_{k_p}^{k} p^2 E_{TKE}(p)\,dp]^{-1/2}$ (labeled as revised with τ_{NL}). In all cases, the flux transfer term is ignored. The vertical lines indicate $kz = 3$ and $kz = 6$, respectively.

V. SUMMARY AND CONCLUSIONS

Whether the "−5/3" scaling exponent in the wall-normal energy spectrum is necessary for the attainment of the "−7/3" cospectral scaling exponent was explored. Field data collected over a lake surface in the atmospheric boundary layer reported here showed that the "−7/3" scaling in the momentum flux cospectrum commences at wavenumbers (around $kz = 3$) that are smaller than those associated with the "−5/3" scaling in the vertical velocity energy spectrum (around $kz = 6$). This finding is not a particular feature of the experiment here and is in agreement with prior bench-mark wind tunnel experiments conducted at very high-Reynolds numbers. Collectively, these experiments suggest that the "−5/3" scaling in the wall-normal energy spectrum cannot be a necessary condition for the attainment of the "−7/3" cospectral scaling.

A cospectral budget (CSB) model was then used to explore plausible mechanisms linking the "−5/3" and "−7/3" scaling exponents in the wall-normal spectrum and vertical momentum flux cospectrum, respectively. Previous studies using the CSB model have demonstrated that the energy spectrum following the "−5/3" scaling was not a sufficient condition for the prevalence of the "−7/3" scaling in the cospectrum. Those studies attributed deviations from the "−7/3" scaling to the flux transfer across scales. In this study, the CSB model, without any revisions, was first used to elucidate the possible effects of flux transfer across scales on the cospectral scaling exponent. For consistency with prior studies, the relaxation time scale in the slow-component of the Rotta model as derived from K41 considerations only was used. The necessary conditions for the occurrence of a "−7/3" scaling were then discussed in the absence of a "−5/3" scaling in the energy spectrum. For a finite flux transfer contribution, the onset of a "−7/3" scaling appears to exist only for a set of restricted conditions, which are unlikely to be satisfied.

As a point of departure from these prior studies, the CSB model was then modified by ignoring entirely the flux transfer term across all scales (as a limiting case) but revisions to the relaxation time scale of the Rotta closure were introduced. Those revisions link the relaxation time scale to the turbulent kinetic energy (TKE) spectrum. The CSB model with the revised time scale, especially the non-local version, reasonably captured the measured cospectrum in the wavenumber range of $3 < kz < 6$, where the "−7/3" cospectral scaling approximately holds but not the "−5/3" spectral scaling in the wall-normal velocity spectrum. Hence, the model-data agreement demonstrated that the earlier onset of the "−7/3" scaling in the cospectrum may be related to the slower than the "−5/3" power-law decay in the TKE spectrum as k crosses over from production to inertial scales. Another interesting feature of the revised cospectral budget model is that it does not require a finite flux transfer term to reproduce the "−7/3" scaling. This explanation differs from previous studies that attributed the entire anomalous cospectral scaling in the inertial subrange to a finite flux transfer term. It is shown here that the pressure-decorrelation term, closed using a relaxation time scale describing the slow (or the linear) part and linked to the TKE spectrum, suffices. However, it certainly does not imply that such a simplification would apply universally. Future investigations are encouraged to examine the relation between spectral and cospectral scaling laws using other data sets, and the role of flux transfer as a function of scales and atmospheric conditions.

More broadly, the mismatch between the "−5/3" and "−7/3" scaling laws for wall-normal energy spectrum and momentum flux cospectrum, respectively, in the vicinity of the crossover from production to inertial scales does not necessarily mean that different types of eddies are contributing to the spectrum and cospectrum. In the vicinity of the crossover from production to inertial scales, the main production term for the vertical velocity spectrum is the pressure-redistribution whereas the main sink term for the cospectrum is the pressure-decorrelation. Deviations from the "−5/3" scaling in the vertical velocity spectrum at scales smaller than those associated with the crossover from production to inertial may be due to a number of phenomena, including the spectral energy transfer term. For the flux cospectrum, however, deviations from the "−7/3" scaling over the same range of scales need not to be linked to the cospectral flux transfer term as shown here. The CSB model offered a *diagnostic* approach to show that a "−7/3" scaling in the cospectrum is plausible in the absence of a "−5/3" in the vertical velocity spectrum and finite flux transfer across scales. However, the CSB model as formulated here cannot be *prognostic* as it offers no budget models for the turbulent kinetic energy or the vertical velocity spectrum, both needed for predicting the cospectral scaling laws.

Since the work here focused on the crossover from production to inertial scales and into the inertial range, it is tempting to interpret the eddy sizes of the CSB model in the context of Townsend's attached eddy hypothesis.[48] Attached eddies are often characterized by $kz < 1$ and are assumed to be the primary eddies responsible for the turbulent momentum flux. Detached eddies do not "sense" the surface and are approximated by their inertial subrange scaling (i.e., $kz \gg 1$). Because the co-spectrum is sufficiently small for $kz \gg 1$, contributions from detached eddies to the overall turbulent stress are minor. Based on the CSB model results, roughly 70% of the momentum flux originates from $kz < 1$ and 30% originates from $kz > 1$ when assuming $k\eta \approx \infty$. The CSB model results also suggest that the transition from attached to detached eddies does not occur at a clear-cut scale $kz = 1$. Instead, the transition may be characterized by a sub-region spanning scales that vary from $kz = 1$ to $kz = 10$, where eddies are gradually attaining their K41 scaling but still contributing to the momentum flux.

[1]A. N. Kolmogorov, "The local structure of turbulence in incompressible viscous fluid for very large Reynolds number," Dokl. Akad. Nauk SSSR **30**, 299 (1941).

[2]J. L. Lumley, "The spectrum of nearly inertial turbulence in a stably stratified fluid," J. Atmos. Sci. **21**, 99–102 (1964).

[3]J. C. Wyngaard and O. R. Coté, "Cospectral similarity in the atmospheric surface layer," Q. J. R. Meteorol. Soc. **98**, 590–603 (1972).

[4]J. C. Kaimal, J. C. Wyngaard, Y. Izumi, and O. R. Coté, "Spectral characteristics of surface-layer turbulence," Q. J. R. Meteorol. Soc. **98**, 563–589 (1972).

[5]S. G. Saddoughi and S. V. Veeravalli, "Local isotropy in turbulent boundary layers at high Reynolds number," J. Fluid Mech. **268**, 333–372 (1994).

[6]X. H. Lee, W. Massman, and B. Law, *Handbook of Micrometeorology: A Guide for Surface Flux Measurement and Analysis* (Springer Science & Business Media, 2006), Vol. 29.

[7]L. Mydlarski and Z. Warhaft, "Passive scalar statistics in high-Péclet-number grid turbulence," J. Fluid Mech. **358**, 135–175 (1998).

[8]W. J. T. Bos, H. Touil, L. Shao, and J. P. Bertoglio, "On the behavior of the velocity-scalar cross correlation spectrum in the inertial range," Phys. Fluids **16**, 3818–3823 (2004).

[9]W. J. T. Bos, H. Touil, and J. P. Bertoglio, "Reynolds number dependency of the scalar flux spectrum in isotropic turbulence with a uniform scalar gradient," Phys. Fluids **17**, 125108–126600 (2005).

[10]G. G. Katul, A. Porporato, C. Manes, and C. Meneveau, "Co-spectrum and mean velocity in turbulent boundary layers," Phys. Fluids **25**, 091702 (2013).

[11]G. G. Katul, D. Li, M. Chamecki, and E. Bou-Zeid, "Mean scalar concentration profile in a sheared and thermally stratified atmospheric surface layer," Phys. Rev. E **87**, 023004 (2013).

[12]G. G. Katul, A. Porporato, S. Shah, and E. Bou-Zeid, "Two phenomenological constants explain similarity laws in stably stratified turbulence," Phys. Rev. E **89**, 023007 (2014).

[13]G. G. Katul, D. Li, H. P. Liu, and S. Assouline, "Deviations from unity of the ratio of the turbulent Schmidt to Prandtl numbers in stratified atmospheric flows over water surfaces," Phys. Rev. Fluids **1**, 034401 (2016).

[14]D. Li, G. G. Katul, and S. Zilitinkevich, "Revisiting the turbulent Prandtl number in an idealized atmospheric surface layer," J. Atmos. Sci. **72**, 2394 (2015).

[15]D. Li, "Revisiting the subgrid-scale Prandtl number for large-eddy simulation," J. Fluid Mech. **802**, R2 (2016).

[16]D. Cava and G. G. Katul, "On the scaling laws of the velocity-scalar cospectra in the canopy sublayer above tall forests," Boundary-Layer Meteorol. **145**, 351–367 (2012).

[17]S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, UK, 2000), p. 771.

[18]R. B. Stull, *An Introduction to Boundary Layer Meteorology* (Kluwer Academic Publishers, Dordrecht, 1988).

[19]N. Vercauteren, E. Bou-Zeid, M. B. Parlange, U. Lemmin, H. Huwald, J. Selker, and C. Meneveau, "Subgrid-scale dynamics for water vapor, heat, and momentum over a lake," Boundary-Layer Meteorol. **128**, 205–228 (2008).

[20]D. Li and E. Bou-Zeid, "Coherent structures and the dissimilarity of turbulent transport of momentum and scalars in the unstable atmospheric surface layer," Boundary-Layer Meteorol. **140**, 243–262 (2011).

[21]D. Li, E. Bou-Zeid, and H. A. R. De Bruin, "Monin–Obukhov similarity functions for the structure parameters of temperature and humidity," Boundary-Layer Meteorol. **145**, 45–67 (2012).

[22]D. Li, G. G. Katul, and E. Bou-Zeid, "Mean velocity and temperature profiles in a sheared diabatic turbulent boundary layer," Phys. Fluids **24**, 105105 (2012).

[23]D. Li, G. G. Katul, and E. Bou-Zeid, "Turbulent energy spectra and cospectra of momentum and heat fluxes in the stable atmospheric surface layer," Boundary-Layer Meteorol. **157**, 1–21 (2015).

[24]D. Li, S. T. Salesky, and T. Banerjee, "Connections between the Ozmidov scale and mean velocity profile in stably stratified atmospheric surface layers," J. Fluid Mech. **797**, R3 (2016).

[25]G. I. Taylor, "The spectrum of turbulence," Proc. R. Soc. A **164**, 476–490 (1938).

[26]K. R. Sreenivasan, "On the universality of the Kolmogorov constant," Phys. Fluids **7**, 2778–2784 (1995).

[27]W. J. T. Bos and J. P. Bertoglio, "Inertial range scaling of scalar flux spectra in uniformly sheared turbulence," Phys. Fluids **19**, 025104 (2007).

[28]G. G. Katul, C. I. Hsieh, and J. Sigmon, "Energy-inertial scale interactions for velocity and temperature in the unstable atmospheric surface layer," Boundary-Layer Meteorol. **82**, 49–80 (1997).

[29]K. S. Choi and J. L. Lumley, "The return to isotropy of homogeneous turbulence," J. Fluid Mech. **436**, 59–84 (2001).

[30]K. A. McColl, G. G. Katul, P. Gentine, and D. Entekhabi, "Mean-velocity profile of smooth channel flow explained by a cospectral budget model with wall-blockage," Phys. Fluids **28**, 035107 (2016).

[31]G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation* (Cambridge University Press, 2006).

[32]S. Corrsin, "Further generalization of Onsager's cascade model for turbulent spectra," Phys. Fluids **7**, 1156 (1964).

[33]W. Heisenberg, "On the theory of statistical and isotropic turbulence," Proc. R. Soc. A **195**, 402–406 (1948).

[34]S. Panchev, *Random Functions and Turbulence* (Pergamon Press, Oxford, 1971), p. 443.

[35]G. Katul and C. R. Chu, "A theoretical and experimental investigation of energy-containing scales in the dynamic sublayer of boundary-layer flows," Boundary-Layer Meteorol. **86**, 279–312 (1998).

[36]L. Danaila and R. A. Antonia, "Spectrum of a passive scalar in moderate Reynolds number for homogeneous isotropic turbulence," Phys. Fluids **21**, 111702 (2009).

[37]A. E. Perry and C. J. Abell, "Scaling laws for pipe-flow turbulence," J. Fluid Mech. **67**, 257–271 (1975).

[38]A. E. Perry and C. J. Abell, "Asymptotic similarity of turbulence structures in smooth- and rough-walled pipes," J. Fluid Mech. **79**, 785–799 (1977).

[39]A. E. Perry, S. Henbest, and M. S. Chong, "A theoretical and experimental study of wall turbulence," J. Fluid Mech. **165**, 163–199 (1986).

[40]A. E. Perry, I. Marusic, and J. D. Li, "Wall turbulence closure based on classical similarity laws and the attached eddy hypothesis," Phys. Fluids **6**, 1024–1035 (1994).

[41]G. G. Katul, C. R. Chu, M. B. Parlange, J. D. Albertson, and T. A.Ortenburger, "Low-wavenumber spectral characteristics of velocity and temperature in the atmospheric surface layer," J. Geophys. Res. **100**, 14243–14255, doi:10.1029/94jd02616 (1995).

[42]G. G. Katul, A. Porporato, and V. Nikora, "Existence of k − 1 power-law scaling in the equilibrium regions of wall-bounded turbulence explained by Heisenberg's eddy viscosity," Phys. Rev. E **86**, 066311 (2012).

[43]M. Calaf, M. Hultmark, H. J. Oldroyd, V. Simeonov, and M. B. Parlange, "Coherent structures and the k − 1 spectral behaviour," Phys. Fluids **25**, 125107 (2013).

[44]T. Banerjee and G. G. Katul, "Logarithmic scaling in the longitudinal velocity variance explained by a spectral budget," Phys. Fluids **25**, 125106 (2013).

[45]T. Banerjee, G. G. Katul, S. T. Salesky, and M. Chamecki, "Revisiting the formulations for the logitudinal velocity variance in the unstable atmospheric surface layer," Q. J. R. Meteorol. Soc. **141**, 1699–1711 (2014).

[46]D. Li, G. G. Katul, and P. Gentine, "The k^{-1} scaling of air temperature spectra in atmospheric surface layer flows," Q. J. R. Meteorol. Soc. **142**, 496 (2016).

[47]P. Drobinski, P. Carlotti, J. L. Redelsperger, V. Masson, R. M. Banta, and R. K. Newsom, "Numerical and experimental investigation of the neutral atmospheric surface layer," J. Atmos. Sci. **64**, 137–156 (2007).

[48]A. A. Townsend, *The Structure of Turbulent Shear Flow* (Cambridge University Press, Cambridge, 1976), Vol. 2.