Name: Prof D   Discussion Section:______

MA 226 Section B – Exam 3a

Spring 2014

Question Number	Possible Points	Student Score
1	16	
2	12	
3	14	
4	12	
5	10	
6	12	
7	10	
8	14	
Total Points	100	

You must show your work to receive full credit

Discussion Sections:

B2: Tuesday 4:30-5:30

B3: Tues : 3:30-4:30

B4: Weds: 9-10

B5: Weds: 10-11

B6: Weds: 4:30-5:30

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1.) (4 pts each) Find the Inverse Laplace Transform of the following expressions:

a) $\mathcal{L}^{-1}\left[\frac{3}{4s-2}\right] = \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s-\frac{1}{2}}\right] = \boxed{\frac{3}{4}e^{\frac{1}{2}t}}$

b) $\mathcal{L}^{-1}\left[\frac{s+3}{s(s^2+3)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s+1}{s^2+3}\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+3}\right] + \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{s^2+3}\right]$

$= \boxed{1 - \cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t}$

$\frac{s+3}{s(s^2+3)} = \frac{A}{s} + \frac{Bs+C}{s^2+3}$

$s+3 = As^2 + 3A + Bs^2 + Cs$

$s+3 = (A+B)s^2 + Cs + 3A$

$3A = 3 \Rightarrow A = 1$

$C = 1$

$B = -1$

check

$\frac{1}{s} + \frac{-s+1}{s^2+3} = \cancel{s^2}+3-\cancel{s^2}+s$ ✓

c) $\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^3}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2e^{-s}}{s^3}\right] = \boxed{\frac{1}{2}u_1(t)(t-1)^2}$

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1.) (continued) Find the inverse Laplace transform:

d) $\mathcal{L}^{-1}\left[\dfrac{s\cdot e^{-3s}}{s^2-4s+6}\right]=$ $\mathcal{L}^{-1}\left[\dfrac{s e^{-3s}}{(s-2)^2+2}\right]$

$$= \mathcal{L}^{-1}\left[\frac{((s-2)+2)e^{-3s}}{(s-2)^2+2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{(s-2)e^{-3s}}{(s-2)^2+2}\right] + \sqrt{2}\,\mathcal{L}^{-1}\left[\frac{\sqrt{2}\,e^{-3s}}{(s-2)^2+2}\right]$$

$$= u_3(t)\,e^{2(t-3)}\cos(\sqrt{2}(t-3)) +$$

$$\sqrt{2}\,u_3(t)\,e^{2(t-3)}\sin(\sqrt{2}(t-3))$$

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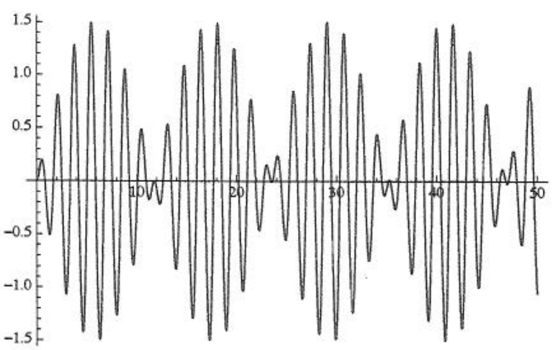
2. (12 pts) Six second order equations and four $y(t)$ - graphs are given below. For each $y(t)$ - graph, determine the second-order equation for which $y(t)$ is a solution, and state briefly how you know your choice is correct.

(i) $\dfrac{d^2y}{dt^2}+18y=3\cos 4t$ (ii) $\dfrac{d^2y}{dt^2}+16y=8$ (iii) $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+16y=3\cos 4t$

(iv) $\dfrac{d^2y}{dt^2}+16y=3\cos 4t$ (v) $\dfrac{d^2y}{dt^2}+16y=-8$ (vi) $\dfrac{d^2y}{dt^2}+12y=3\cos 4t$ $F_N = \frac{2\sqrt{3}}{2\pi}$ $F_f = \frac{4}{2\pi}$

Beating freq = $\left(\frac{F_N - F_f}{2}\right) \approx$

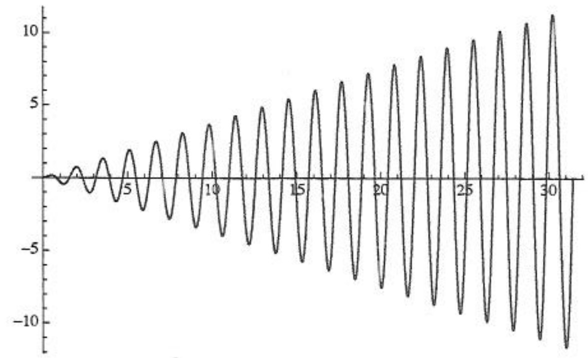
A Beat period is $\approx$ 24



equation: vi

reason: Beat period is $\approx$ 24

B



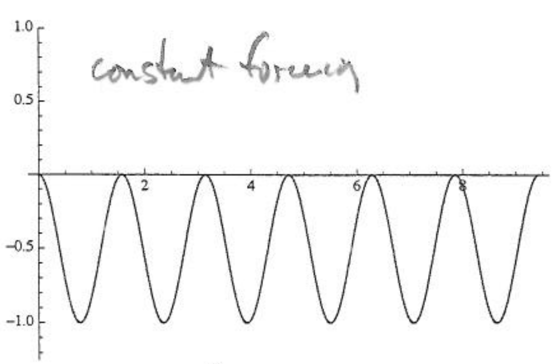
equation: iv

reason: resonance frequency

Forcing freq = natural frequency

C

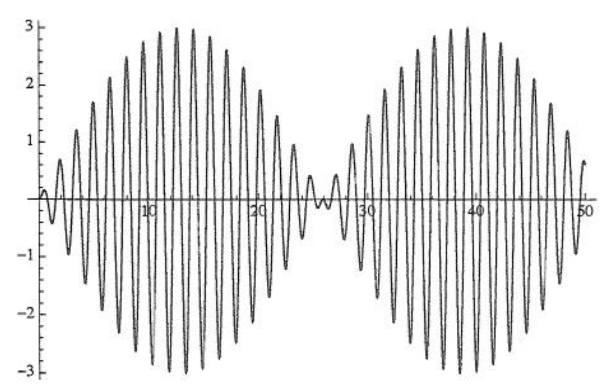
constant forcing



equation: ✓

reason: constant negative forcing

D Beat period $\approx$ 50



equation: (i)

reason: Beat Period $\approx$ 50

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3. (14 pts) Find the general solution of : $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+8y=16t+e^{-2t}$

characteristic eqn $s^2+6s+8=0$

$s=\frac{-6\pm\sqrt{36-32}}{2}=\frac{-6\pm 2}{2}$ $s=-2,-4$

4pt $\boxed{y_h(t)=k_1e^{-2t}+k_2e^{-4t}}$

$y_p(t)=At+B$

$y_p'(t)=A$

$y_p''(t)=0$

$0+6A+8(At+B)=16t$

$8At+(6A+8B)=16t$

$8A=16$ $\boxed{A=2}$

$6A+8B=0$ $\boxed{B=-\frac{12}{8}=-\frac{3}{2}}$

5pts $\boxed{y_{p_1}(t)=2t-\frac{3}{2}}$

$y_{p_2}(t)=Ate^{-2t}$

$y_{p_2}'(t)=Ae^{-2t}-2Ate^{-2t}$

$y_{p_2}''(t)=-2Ae^{-2t}-2Ae^{-2t}+4Ate^{-2t}$

$-4Ae^{-2t}+4Ate^{-2t}+6(Ae^{-2t}-2Ate^{-2t})+8Ate^{-2t}=e^{-2t}$

$(-4A+6A)Ae^{-2t}+\cancel{12Ate^{-2t}}-\cancel{12Ate^{-2t}}=1\cdot e^{-2t}$

$2A=1 \Rightarrow A=\frac{1}{2}$

5pts $\boxed{y_{p_2}(t)=\frac{1}{2}te^{-2t}}$

General Solution

$$\boxed{\begin{aligned} y(t)&=y_h(t)+y_{p_1}(t)+y_{p_2}(t)\\ y(t)&=k_1e^{-2t}+k_2e^{-4t}+2t-\frac{3}{2}+\frac{1}{2}te^{-2t}\end{aligned}}$$

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4. Short answer (4 pts each)

a) What is the beating frequency and the rapid response frequency of the solution to the undamped equation: $\frac{d^2y}{dt^2}+9y=2\cos(4t)$

$F_N = \frac{3}{2\pi} \qquad F_f = \frac{4}{2\pi}$

$F_B = \left|\frac{F_N - F_f}{2}\right| = \frac{1}{4\pi}$

$F_R = \frac{F_N + F_f}{2} = \frac{7}{4\pi}$

b) Given the 2nd order linear non-homogeneous differential equation $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+13y=3\cos(2t)$ has general solution $y(t)=k_1e^{-2t}\cos 3t+k_2e^{-2t}\sin 3t+\underbrace{\frac{27}{145}\cos 2t+\frac{24}{145}\sin 2t}_{\text{multiply by 5}}$. What is the general solution of $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+13y=15\cos(2t)$

$y(t) = k_1 e^{-2t}\cos 3t + k_2 e^{-2t}\sin 3t + \frac{27}{29}\cos 2t + \frac{24}{29}\sin 2t$

c) What is the amplitude of the steady state behavior of behavior of a 2nd order differential equation whose general solution is given by : $y(t)=k_1e^{-t}+k_2e^{-3t}+\underbrace{3\sin 2t+5\cos 2t}_{\text{steady state behavior}}$

$A = \sqrt{3^2+5^2} = \sqrt{34}$

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5. (10 pts) Compute the Laplace Transform of the function $w(t)$ using the definition of the Laplace Transform:

$$w(t)=\begin{cases} 2t, \text{ if } 0\le t<1 \\ 2, \quad \text{if } t\ge 1 \end{cases}$$

$$\mathcal{L}\{w(t)\} = \int_0^\infty w(t)e^{-st}\,dt$$

$$= \int_0^1 \underbrace{2t}_{u}\underbrace{e^{-st}\,dt}_{dv} + \int_1^\infty 2e^{-st}\,dt$$

$du = 2dt \quad v = \frac{e^{-st}}{-s}$

5 pts

$$= 2t\left(\frac{-e^{-st}}{s}\right)\Big|_0^1 - \int_0^1 -\frac{e^{-st}}{s}\cdot 2\,dt + \lim_{b\to\infty} 2\int_1^b e^{-st}\,dt$$

5 pts

$$= \frac{-2e^{-s}}{s} - 0 - \frac{2}{s^2}e^{-st}\Big|_0^1 + \lim_{b\to\infty} -\frac{2}{s}e^{-st}\Big|_1^b$$

$$= \cancel{-\frac{2e^{-s}}{s}} - \frac{2}{s^2}e^{-s} + \frac{2}{s^2} + \lim_{b\to\infty} -\frac{2}{s}e^{-sb} \to 0 + \cancel{\frac{2}{s}e^{-s}}$$

$$= \frac{2}{s^2} - \frac{2}{s^2}e^{-s}$$

$$= \frac{2}{s^2}\left(1-e^{-s}\right)$$

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7. (10 pts) The nonlinear autonomous system

$$\frac{dx}{dt} = x(2-x-y) = 2x - x^2 - xy = f(x,y)$$
$$\frac{dy}{dt} = y(y-x^2) = y^2 - x^2 y = g(x,y)$$

has an equilibrium point at $(-2,4)$. Use Linearization to classify they type of equilibrium point. If the equilibrium point is a spiral sink or a spiral source give the direction of the spiral.

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 2-2x-y & -x \\ -2xy & 2y-x^2 \end{pmatrix}$$ 3pts

$$J(-2,4) = \begin{pmatrix} 2 & 2 \\ 16 & 4 \end{pmatrix}$$ 2pts

Linear System $\frac{d\bar{y}}{dt} = \begin{bmatrix} 2 & 2 \\ 16 & 4 \end{bmatrix} \bar{y}$

eigenvalues $\det \begin{bmatrix} 2-\lambda & 2 \\ 16 & 4-\lambda \end{bmatrix} = 0$

$$(2-\lambda)(4-\lambda) - 32 = 0$$
$$\lambda^2 - 6\lambda + 8 - 32 = 0$$
$$\lambda^2 - 6\lambda - 24 = 0$$
$$\boxed{\lambda = \frac{6 \pm \sqrt{36 - 4\cdot(-24)}}{2} = 6 \pm \sqrt{33}}$$ 3pts

$\lambda_1 = 3 + \sqrt{33} > 0$

$\lambda_2 = 3 - \sqrt{33} < 0$ $\Rightarrow$ saddle point (2pts)

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6. (12 pts) Find a particular solution of $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+8y=5\cos 3t$ using complexification.

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+8y = 5e^{3it}$$

$y_c(t) = Ae^{3it}$

$y_c'(t) = 3iAe^{3it}$

$y_c''(t) = -9Ae^{3it}$

plug in

$$-9Ae^{3it} + 18iAe^{3it} + 8Ae^{3it} = 5e^{3it}$$

$$(-A + 18iA)e^{3it} = 5e^{3it}$$

$$A(-1+18i) = 5$$

$$A = \frac{5}{-1+18i} = \frac{5}{325}(-1-18i)$$

$$\boxed{A = \frac{-1}{65}(1+18i)}$$

6 pts

$$y_c(t) = \frac{-1}{65}(1+18i)(\cos 3t + i\sin 3t)$$

$$= \frac{-1}{65}\Big(\underbrace{(\cos 3t - 18\sin 3t)}_{\text{Re}(y_c(t))} + i\,(18\cos 3t + \sin 3t)\Big)$$

$$\boxed{y_p(t) = -\frac{1}{65}\cos 3t + \frac{18}{65}\sin 3t}$$

(6 pts)

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8. (14 pts) Solve the 2nd order linear non-homogeneous initial value problem:

(you can use the next page too if you need more space)

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+13y=13u_4(t) \quad \text{with } y(0)=1 \text{ and } y'(0)=2$$

$$s^2\mathcal{L}[y] - s\,y(0) - y'(0) + 6\left[s\mathcal{L}[y] - y(0)\right] + 13\mathcal{L}[y] = \mathcal{L}\left[13u_4(t)\right]$$

$$s^2\mathcal{L}[y] - s\cdot 1 - 2 + 6s\mathcal{L}[y] - 6 + 13\mathcal{L}[y] = 13\frac{e^{-4s}}{s}$$

$$(s^2+6s+13)\mathcal{L}[y] - s - 8 = 13\frac{e^{-4s}}{s}$$

5 pts
$$\boxed{\mathcal{L}[y] = \frac{s+8}{s^2+6s+13} + \frac{13e^{-4s}}{s(s^2+6s+13)}}$$

partial fractions 3pts (next page)

$$\mathcal{L}[y] = \frac{s+8}{(s+3)^2+4} + e^{-4s}\left(\frac{1}{s} - \frac{s+6}{s^2+6s+13}\right)$$

$$= \frac{s+3+5}{(s+3)^2+4} + e^{-4s}\left(\frac{1}{s} - \frac{s+3+3}{(s+3)^2+4}\right)$$

$$\mathcal{L}[y] = \frac{s+3}{(s+3)^2+4} + \frac{5}{2}\frac{2}{(s+3)^2+4} + \frac{e^{-4s}}{s} - \frac{(s+3)e^{-4s}}{(s+3)^2+2^2} + \frac{3}{2}\frac{2\cdot e^{-4s}}{(s+3)^2+4}$$

$$y(t) = \overbrace{e^{-3t}\cos 2t + \frac{5}{2}e^{-3t}\sin 2t}^{2\text{ pts}} + u_4(t) - u_4(t)e^{-3(t-4)}\cos 2(t-4) + \frac{3}{2}u_4(t)e^{-3(t-4)}\sin 2(t-4)$$

4 pts

$$\frac{13}{s(s^2+6s+13)} = \frac{A}{s} + \frac{Bs+C}{s^2+6s+13}$$

$$\frac{As^2+6As+13A+Bs^2+Cs}{s(s^2+6s+13)}$$

$$13 = (A+B)s^2 + (6A+C)s + 13A$$

$$A+B = 0$$
$$6A+C = 0$$
$$13A = 13 \Rightarrow A=1, \quad B=-1 \quad C=-6$$

Table 6.1

Frequently Encountered Laplace Transforms.

$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$	$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a} \quad (s > a)$	$y(t) = t^n$	$Y(s) = \dfrac{n!}{s^{n+1}} \quad (s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \dfrac{\omega}{s^2 + \omega^2}$	$y(t) = \cos \omega t$	$Y(s) = \dfrac{s}{s^2 + \omega^2}$
$y(t) = e^{at} \sin \omega t$	$Y(s) = \dfrac{\omega}{(s-a)^2 + \omega^2}$	$y(t) = e^{at} \cos \omega t$	$Y(s) = \dfrac{s-a}{(s-a)^2 + \omega^2}$
$y(t) = t \sin \omega t$	$Y(s) = \dfrac{2\omega s}{(s^2 + \omega^2)^2}$	$y(t) = t \cos \omega t$	$Y(s) = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s} \quad (s > 0)$	$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$

Table 6.2

Rules for Laplace Transforms:

Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants α and a.

Rule for Laplace Transform	Rule for Inverse Laplace Transform
$\mathcal{L}\left[\dfrac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) = sY(s) - y(0)$	
$\mathcal{L}[y + w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$	$\mathcal{L}^{-1}[Y + W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$
$\mathcal{L}[\alpha y] = \alpha \mathcal{L}[y] = \alpha Y(s)$	$\mathcal{L}^{-1}[\alpha Y] = \alpha \mathcal{L}^{-1}[Y] = \alpha y(t)$
$\mathcal{L}[u_a(t) y(t-a)] = e^{-as} \mathcal{L}[y] = e^{-as} Y(s)$	$\mathcal{L}^{-1}[e^{-as} Y] = u_a(t) y(t-a)$
$\mathcal{L}[e^{at} y(t)] = Y(s-a)$	$\mathcal{L}^{-1}[Y(s-a)] = e^{at} \mathcal{L}^{-1}[Y] = e^{at} y(t)$