Name: Pat D

Discussion Section:______

# MA 226 Section B – Exam 3a

# Fall 2014

| Question Number | Possible Points | Student Score |
|---|---|---|
| 1 | 24 | |
| 2 | 12 | |
| 3 | 14 | |
| 4 | 10 | |
| 5 | 12 | |
| 6 | 12 | |
| 7 | 16 | |
| Total Points | 100 | |

## You must show your work to receive full credit

Discussion Sections:

B2: Wednesday 9-10

B3: Wednesday 2-3

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1.) Find the Inverse Laplace Transform of the following expressions:

a) $\mathcal{L}^{-1}\left[\frac{6}{s^2-1}\right] = \mathcal{L}^{-1}\left[\frac{6}{(s-1)(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{3}{s-1} - \frac{3}{s+1}\right] = \boxed{3e^{t} - 3e^{-t}}$

(6 pts)

$$\frac{6}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{A(s+1) + B(s-1)}{(s-1)(s+1)}$$

$$6 = (A+B)s + (A-B)$$

$$0 = A+B$$
$$6 = A-B$$
$$2A = 6 \quad \boxed{A=3} \quad \boxed{B=-3}$$

b) $\mathcal{L}^{-1}\left[\frac{2s}{s^3(s-1)}\right] = \mathcal{L}^{-1}\left[\frac{-2s^2-2s}{s^3} + \frac{2}{s-1}\right] = \mathcal{L}^{-1}\left[-\frac{2}{s}\right] - \mathcal{L}^{-1}\left[\frac{2}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{s-1}\right]$

(8 pts)

$$\frac{2s}{s^3(s-1)} = \frac{As^2+Bs+C}{s^3} + \frac{D}{s-1}$$

$$= \frac{As^3 + Bs^2 + Cs - As^2 - Bs - C + s^3D}{s^3(s-1)}$$

$$2s = (A+D)s^3 + (B-A)s^2 + (C-B)s - C$$

$\Rightarrow$
$$A+D=0 \qquad D=2$$
$$B-A=0 \qquad A=-2$$
$$C-B=2 \Rightarrow B=-2$$
$$C=0$$

$$\boxed{-2-2t+2e^{t}}$$

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1.) (continued) Find the inverse Laplace transform:

c) $\mathcal{L}^{-1}\left[\frac{s\cdot e^{-2s}}{s^2+9}\right]=$ $\boxed{u_2(t)\cos(3(t-2))}$ (4 pts)

d) $\mathcal{L}^{-1}\left[\frac{(s+1)\cdot e^{-4s}}{s^2+6s+13}\right]=\mathcal{L}^{-1}\left[e^{-4s}\cdot\frac{s+1}{(s+3)^2+4}\right]$ (6 pts)

$$=\mathcal{L}^{-1}\left[e^{-4s}\cdot\frac{s+1+2-2}{(s+3)^2+4}\right]$$

$$=\mathcal{L}^{-1}\left[e^{-4s}\cdot\frac{s+3}{(s+3)^2+4}\right]-\mathcal{L}^{-1}\left[e^{-4s}\cdot\frac{2}{(s+3)^2+4}\right]$$

$$=u_4(t)\,e^{-3(t-4)}\cos(2(t-4))-u_4(t)\,e^{-3(t-4)}\sin(2(t-4))$$

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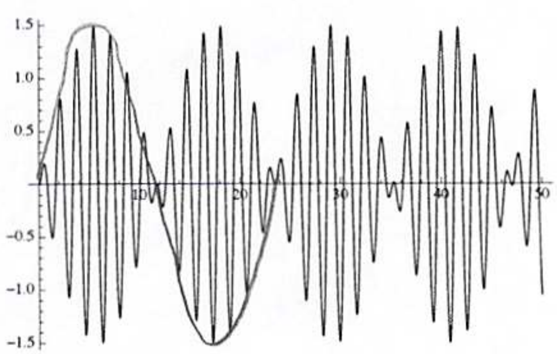
2. (12 pts) Six second order equations and four $y(t)$ - graphs are given below. For each $y(t)$ - graph, determine the second-order equation for which $y(t)$ is a solution, and state briefly how you know your choice is correct.

(i) $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+16y=3\cos 4t$ (ii) $\dfrac{d^2y}{dt^2}+12y=3\cos 4t$ (iii) $\dfrac{d^2y}{dt^2}+18y=3\cos 4t$

(iv) $\dfrac{d^2y}{dt^2}+16y=-8$ (v) $\dfrac{d^2y}{dt^2}+16y=3\cos 4t$ (vi) $\dfrac{d^2y}{dt^2}+16y=8$

A



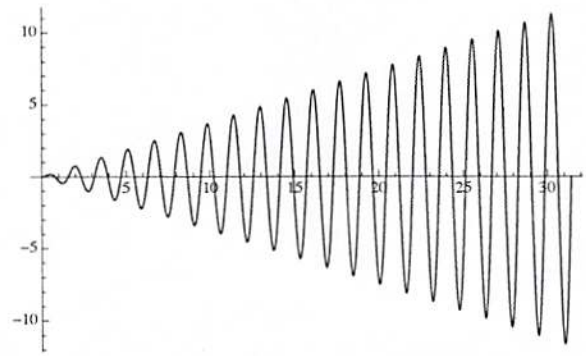

equation: ii

reason: Beating Period $\approx 25$

Beating Freq $\approx \frac{1}{25} \approx .04$

$F_{Beat} = \frac{4-\sqrt{12}}{4\pi} \approx .042$

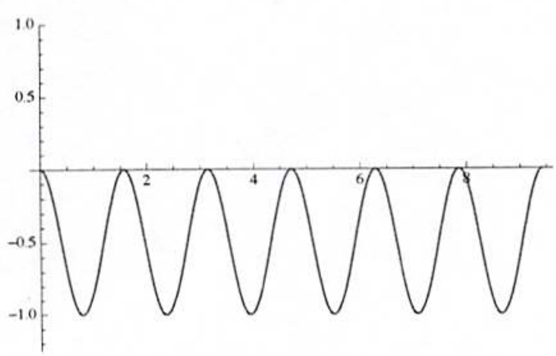
B




equation: V

reason: $F_N = F_f = \frac{4}{2\pi}$

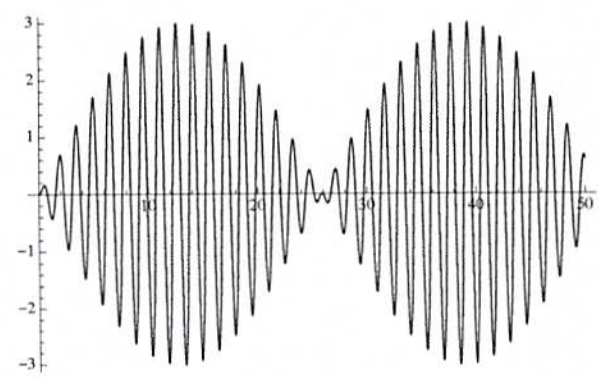
C




equation: iV

reason: forcing function negative constant

D




equation: iii

reason: Beating Period $\approx 50$

Beating Freq $\approx \frac{1}{50} = .02$

$F_{Beat} = \frac{\sqrt{18}-4}{4\pi} = .019$

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3. (14 pts) Find the general solution of : $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=4t^2+2+e^{-2t}$

① $y_h(t) \Rightarrow y(t)=e^{st}$ $\quad s^2+3s+2=0$

$(s+2)(s+1)=0 \Rightarrow s=-2,\ s=-1$

$y_h(t)=k_1e^{-2t}+k_2e^{-t}$ 3pts

② $y_{p_1}(t)=At^2+Bt+C$

$y_{p_1}'(t)=2At+B$

$y_{p_1}''(t)=2A$

6pts

$2A+3(2At+B)+2(At^2+Bt+C)=4t^2+2$

$2At^2+(6A+2B)t+(2A+3B+2C)=4t^2+2$

$2A=4 \Rightarrow \boxed{A=2}$

$6A+2B=0 \Rightarrow \boxed{B=-3A=-6}$

$2A+3B+2C=2 \Rightarrow 4-18+2C=2 \Rightarrow \boxed{C=8}$

$y_{p_1}(t)=2t^2-6t+8$

$y_{p_2}(t)=Ate^{-2t}$

$y_{p_2}'(t)=-2Ate^{-2t}+Ae^{-2t}$

$y_{p_2}''(t)=4Ate^{-2t}-2Ae^{-2t}-2Ae^{-2t}$

plug into eqn

$4Ate^{-2t}-4Ae^{-2t}+3(-2Ate^{-2t}+Ae^{-2t})+2Ate^{-2t}=e^{-2t}$

$\cancel{4Ate^{-2t}}\ \cancel{-6Ate^{-2t}}\ \cancel{+2Ate^{-2t}}-4Ae^{-2t}+3Ae^{-2t}=e^{-2t}$

5pts

$-Ae^{-2t}=e^{-2t} \Rightarrow A=-1$

$y_{p_2}(t)=te^{-2t}$

$y(t)=k_1e^{-2t}+k_2e^{-t}+2t^2-6t+8-te^{-2t}$

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4. (10 pts) Compute the Laplace Transform of the function $w(t)$ using the definition of the Laplace Transform:

$$w(t)=\begin{cases} e^{-t}, \text{ if } 0\le t<1 \\ 3, \quad \text{ if } t\ge 1 \end{cases}$$

$$\mathcal{L}\{w(t)\} = \int_0^\infty w(t)\, e^{-st}\, dt = \int_0^1 e^{-t} e^{-st} dt + \int_1^\infty 3 e^{-st} dt$$

$$= \int_0^1 e^{-t(s+1)} dt + \lim_{b\to\infty} 3\int_1^b e^{-st} dt$$

$$= \frac{-1}{s+1} e^{-t(s+1)} \Big|_0^1 + \lim_{b\to\infty} \frac{-3}{s} e^{-st} \Big|_1^b$$

$$= -\frac{1}{(s+1)}\left(e^{-(s+1)} - e^{0}\right) + \lim_{b\to\infty} -\frac{3}{s}\left(e^{-sb} - e^{-s}\right)$$

$$= \boxed{\frac{1}{s+1}\left(1 - e^{-(s+1)}\right) + \frac{3}{s} e^{-s}}$$

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5. (12 pts) Find a particular solution of $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+13y=3\sin(2t)$ using complexification.

$\underbrace{3\sin(2t)}_{3e^{2ti}}$

Let $y_p(t) = Ae^{2ti}$

$y_p'(t) = 2iAe^{2ti}$

$y_p''(t) = -4Ae^{2ti}$

plug into eqn

$$-4Ae^{2ti} + 4\cdot 2iAe^{2ti} + 13Ae^{2ti} = 3e^{2ti}$$

$$(9A + 8iA)e^{2ti} = 3e^{2ti}$$

$$(9+8i)A = 3$$

$$A = \frac{3}{9+8i} = \frac{3}{9+8i}\cdot\frac{9-8i}{9-8i} = \frac{27-24i}{145}$$

$$A = \frac{3}{145}(9-8i)$$

$$y_p(t) = Ae^{2it} = \frac{3}{145}(9-8i)(\cos 2t + i\sin 2t)$$

$$= \frac{3}{145}\left((9\cos 2t + 8\sin 2t) + i\,(9\sin(2t) - 8\cos(2t))\right)$$

We need the imaginary part

$$y_p(t) = \frac{3}{145}\left(9\sin(2t) - 8\cos(2t)\right)$$

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6. (12 pts) Given the nonlinear autonomous system

$$\frac{dx}{dt} = x(2-x-y) = 2x - x^2 - xy = f(x,y)$$
$$\frac{dy}{dt} = y(6-2x-y) = 6y - 2xy - y^2 = g(x,y)$$

Use Linearization to classify the behavior at the equilibrium point $(4,-2)$. If the equilibrium point is a spiral source or a spiral sink determine the direction of rotation as clockwise or counter clockwise.

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2-2x-y & -x \\ -2y & 6-2x-2y \end{bmatrix}$$

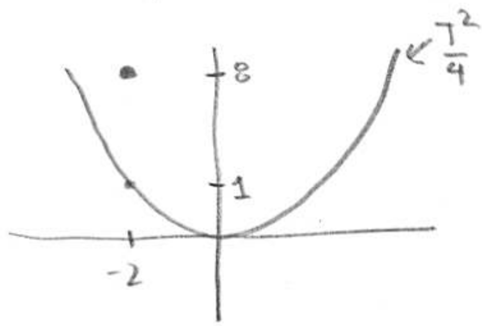
$$J(4,-2) = \begin{bmatrix} 2-8+2 & -4 \\ 4 & 6-8+4 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 4 & 2 \end{bmatrix}$$

Linearized system at $(4,-2)$

$$\frac{d\bar{y}}{dt} = J(4-2)\bar{y} = \begin{bmatrix} -4 & -4 \\ 4 & 2 \end{bmatrix} \bar{y}$$

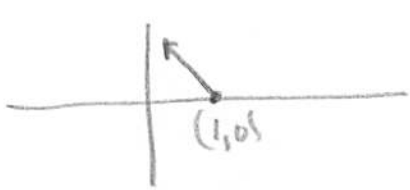
Trace $= -2$

Det $= -8 + 16 = 8$




$\boxed{\text{spiral sink}}$

$$\begin{bmatrix} -4 & -4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$ y is positive $\Rightarrow$ counter clockwise rotation



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7. (16 pts) Solve the 2nd order linear non-homogeneous initial value problem:

(you can use the next page too if you need more space)

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+6y=6u_3(t) \text{ with } y(0)=2 \text{ and } y'(0)=4$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}+4\frac{dy}{dt}+6y\right]=\mathcal{L}[6u_3(t)]$$

$$s^2\mathcal{L}[y]-sy(0)-y'(0)+4[s\mathcal{L}[y]-y(0)]+6\mathcal{L}[y]=\frac{6e^{-3s}}{s}$$

$$(s^2+4s+6)\mathcal{L}[y]-2s-4-8=\frac{6e^{-3s}}{s}$$

$$\mathcal{L}[y]=\frac{2s+12}{s^2+4s+6}+e^{-3s}\cdot\frac{6}{s(s^2+4s+6)}$$

Two separate problem.

$$\mathcal{L}^{-1}\left[\frac{2s+12}{s^2+4s+6}\right]=2\mathcal{L}^{-1}\left[\frac{s+6}{(s+2)^2+2}\right]=2\mathcal{L}^{-1}\left[\frac{s+2+4}{(s+2)^2+2}\right]$$

$$=2\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+2}\right]+2\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2+2}\right]$$

$$=2\cdot e^{-2t}\cos(\sqrt{2}t)+\frac{8}{\sqrt{2}}\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s+2)^2+2}\right]$$

$$=2\cdot e^{-2t}\cos(\sqrt{2}t)+4\sqrt{2}\,e^{-2t}\sin(\sqrt{2}t)$$

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7) $\mathcal{L}^{-1}\left[e^{-3s}\cdot\frac{6}{s(s^2+4s+6)}\right]$

$$\frac{6}{s(s^2+4s+6)} = \frac{A}{s}+\frac{Bs+C}{s^2+4s+6} = \frac{As^2+4As+6A+Bs^2+Cs}{s(s^2+4s+6)}$$

$$\Rightarrow (A+B)s^2+(4A+C)s+6A = 6$$

$$A=1 \Rightarrow B=-1 \Rightarrow C=-4$$

$$\mathcal{L}^{-1}\left[e^{-3s}\cdot\left(\frac{1}{s}-\frac{s+4}{s^2+4s+6}\right)\right] = \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right]-\mathcal{L}^{-1}\left[\frac{(s+4)e^{-3s}}{(s+2)^2+2}\right]$$

$$= u_3(t)-\mathcal{L}^{-1}\left[\frac{(s+2+2)e^{-3s}}{(s+2)^2+2}\right]$$

$$= u_3(t)-\mathcal{L}^{-1}\left[e^{-3s}\cdot\frac{s+2}{(s+2)^2+2}\right]-\sqrt{2}\,\mathcal{L}^{-1}\left[e^{-3s}\cdot\frac{\sqrt{2}}{(s+2)^2+2}\right]$$

$$= u_3(t)-u_3(t)e^{-2(t-3)}\cos(\sqrt{2}(t-3))-\sqrt{2}\,u_3(t)e^{-2(t-3)}\sin(\sqrt{2}(t-3))$$

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$$y(t) = 2e^{-2t}\left(\cos\sqrt{2}t+2\sqrt{2}\sin\sqrt{2}t\right)+$$

$$u_3(t)\left(1-e^{-2(t-3)}\left(\cos(\sqrt{2}(t-3))+\sqrt{2}\sin(\sqrt{2}(t-3))\right)\right)$$