

Name: Prof D

Discussion Section : _____

MA 226 Section B – Exam 2b

Question Number	Possible Points	Student Score
1	15	
2a	5	
2b	5	
2c	5	
2d	5	
2e	5	
3	10	
4	8	
5	12	
6	8	
7	10	
8	12	
Total Points	100	

You must show your work to receive full credit

Discussion Sections:

B2: Wednesday 9-10

B3: Wednesday 2-3

1. Short Answer (15 pts)

- a) Characterize the behavior of the harmonic oscillator with mass $m = 1$, spring constant $k = 5$ and damping coefficient $b = 6$, as underdamped, overdamped or critically damped. Show the work that justifies your answer.

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = 0$$

$$s^2 + 6s + 5 = 0$$

$$s = \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$s = -3 \pm 1 = -4 \text{ or } -2$$

$\therefore$ overdamped

$$b^2 - 4k > 0$$

reason 3 points

- b) Find all the equilibrium solutions of the system of differential equations

$$\frac{dx}{dt} = 3\left(1 - \frac{x}{3}\right)x - xy$$

$$\frac{dy}{dt} = -16y + 8xy$$

$$x(3 - x - y) = 0$$

$$y(-16 + 8x) = 0$$

$x = 0, y = 0$	1 pt
$x = 3, y = 0$	2 pt
$x = 2, y = 1$	2 pt

c) Consider the two systems of differential equations

(i)

$$\frac{dx}{dt} = 0.4x - 0.2xy$$

$$\frac{dy}{dt} = -0.2y + 3xy$$

(ii)

$$\frac{dx}{dt} = 0.4x - 4xy$$

$$\frac{dy}{dt} = -3y + 0.1xy$$

One of these systems refers to a predator-prey system with very lethargic predators- those who seldom catch prey but who can live for a long time on a single prey (for example, boa constrictors). The other system refers to a very active predator that requires many prey to stay healthy (such as a small cat). The prey in each case is the same. Identify which system is which and justify your answer.

(i) lethargic predators because it has a small effect on the prey population. ($|-0.2xy|$ term is smaller than $|-4xy|$)

(ii) active predators. large effect on the prey
 $|-4xy|$ term is $>$ than $|-0.2xy|$ term

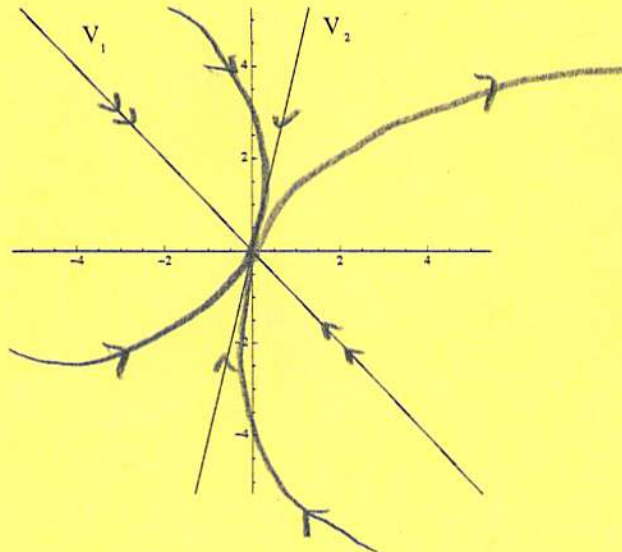
Reason is 3pts

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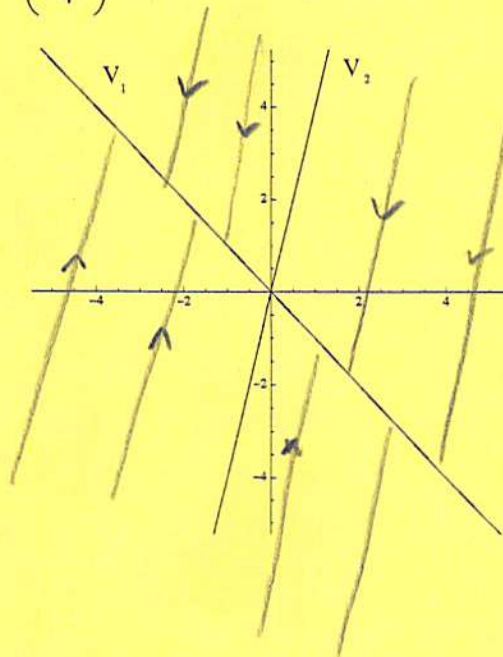
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2. (5 pts each) Draw the phase portrait of a linear system given the following information:

a.) Given eigenvalues $\lambda_1 = -5$ and $\lambda_2 = -2$ with corresponding eigenvectors $\vec{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\vec{V}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$



b.) Given eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$ with corresponding eigenvectors $\vec{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\vec{V}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

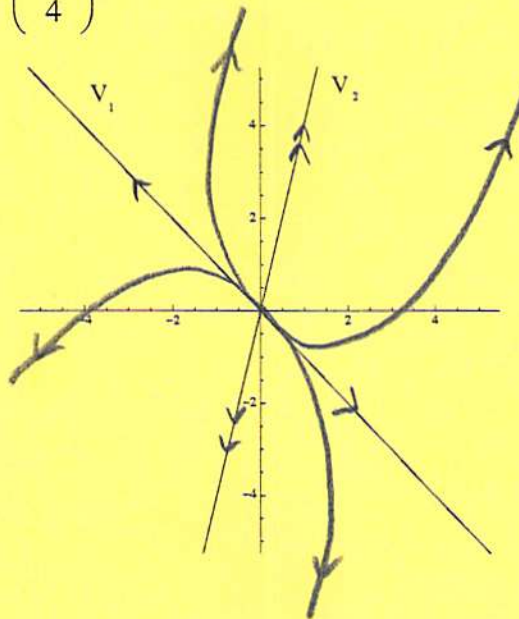


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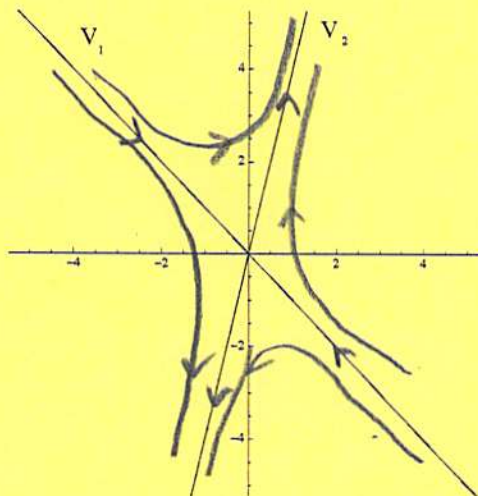
c.) Given eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$ with corresponding eigenvectors

$$\vec{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \vec{V}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$



d.) Given eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$ with corresponding eigenvectors

$$\vec{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \vec{V}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$



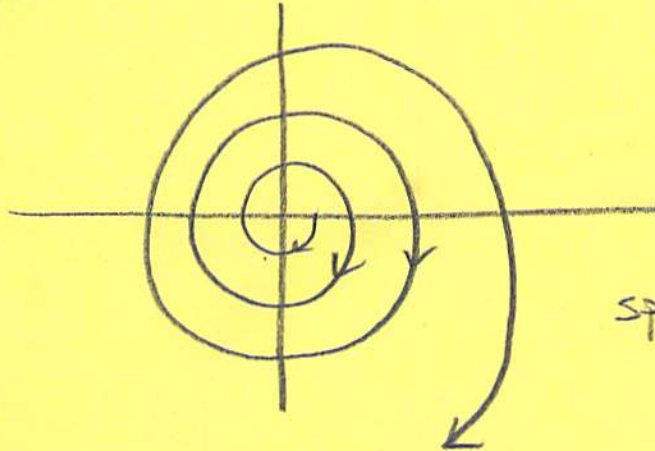
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e.) Draw the phase portrait for the linear system $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \vec{Y}$ with complex eigenvalue $\lambda = 4 + 2i$. Please show how you determined the correct orientation.

spiral source $\text{Re}(\lambda) > 0$ $\begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ negative y

2 pts 3 pts



spiral source.

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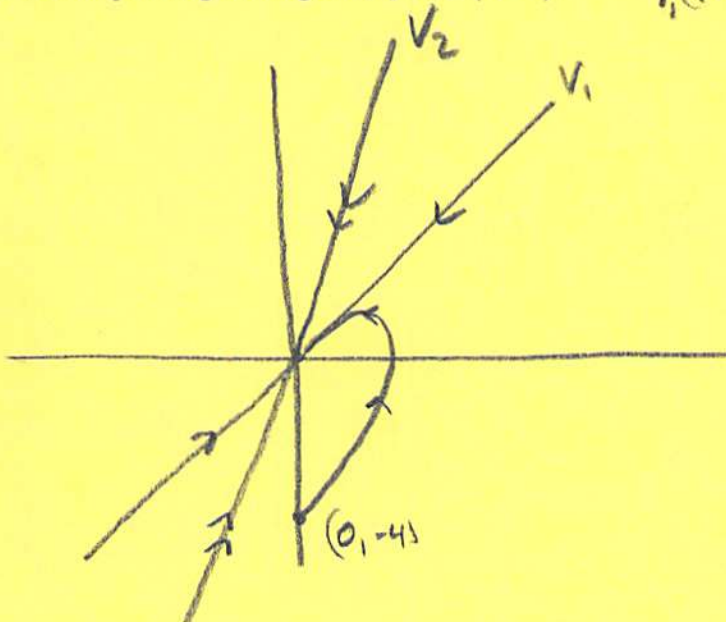
3. (a) (5 pts) Given the linear system $\frac{d\bar{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \bar{Y}$ with straight line solutions

$\bar{Y}_1(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix}$ and $\bar{Y}_2(t) = \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix}$. Make a rough sketch of the solution of the

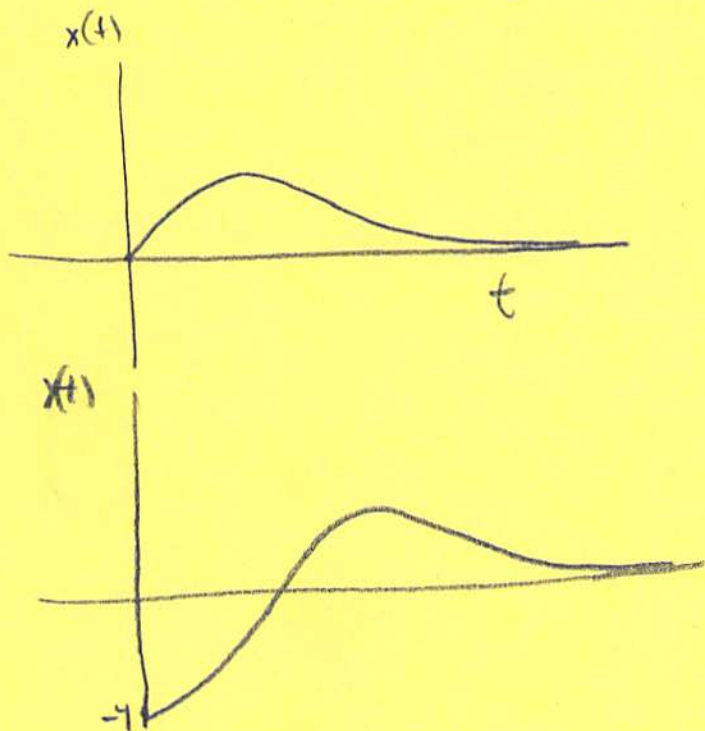
initial value problem passing through the point $(0, -4)$.

$$\bar{Y}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{Y}_2(t) = e^{-4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



(b) (5 pts) Draw the corresponding $x(t)$ and $y(t)$ graphs



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4. (8 pts) Use Euler's Method with a step size of .25 to approximate the solution of the initial value problem at time $t = .75$. Use 6 decimal places of accuracy.

$$\frac{dx}{dt} = -2x + y = f(x, y)$$

where at $t = 0$, we have $(x_0, y_0) = (1, 2)$

$$\frac{dy}{dt} = x = g(x, y)$$

K	t_k	x_k	y_k	$-2x_k + y_k$ $f(x_k, y_k)$	x $g(x_k, y_k)$	x_{k+1}	y_{k+1}
0	$t_0 = 0$	1	2	0	1	1	2.25
1	.25	1	2.25	.25	1	1.0625	2.5
2	.50	1.0625	2.5	.375	1.0625	1.15625	2.765625
3	.75	1.15625	2.765625				

1 pt each

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5. Solve the initial value problem: $\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \bar{Y}$ with initial condition

$$\bar{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

a. (7 pts) Write the general solution.

eigenvalues $\det \begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) + 1 = 0$
 $\lambda^2 - 6\lambda + 9 = 0$
 $(\lambda-3)(\lambda-3) = 0 \Rightarrow \boxed{\lambda=3}$ repeated 2pts

eigenvector $A\bar{v} = 3\bar{v}$

$$2x + y = 3x \Rightarrow y = x$$

$$-x + 4y = 3y \Rightarrow y = x$$

$$\bar{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

General Solution

$$\bar{Y}(t) = e^{3t} \bar{v}_0 + t e^{3t} \bar{v}_1 = e^{3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{3t} \begin{pmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{pmatrix}$$

$$\bar{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

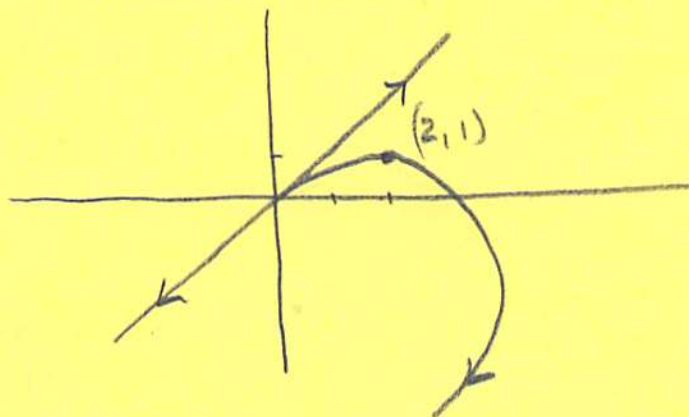
$$\bar{v}_1 = (A - 3I) \bar{v}_0$$

$$= \left(\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{pmatrix}$$
 3pts

b. (2 pts) Write the solution of the initial value problem.

$$\bar{X}(t) = e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t e^{3t} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

c. (3 pts) Make a rough sketch of the solution in the phase plane.



$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

negative y
2pts

6. (8 pts) The linear system $\frac{d\bar{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \bar{Y}$ has complex eigenvalues.

a.) Compute the eigenvalues

$$\det \begin{pmatrix} -\lambda & 1 \\ -5 & -4-\lambda \end{pmatrix} = (-\lambda)(-4-\lambda) + 5 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

4 pts

$$\lambda = \frac{-4 \pm \sqrt{16 - 4 \cdot 5}}{2} = \frac{-4 \pm \sqrt{4}i}{2}$$

$$\lambda = -2 \pm i$$

b.) Compute one eigenvector

$$A\bar{V} = (-2+i)\bar{V}$$

$$\textcircled{1} \quad 0x + y = (-2+i)x \Rightarrow y = (-2+i)x \Rightarrow \bar{V} = \begin{pmatrix} x \\ (-2+i)x \end{pmatrix} = \begin{pmatrix} 1 \\ -2+i \end{pmatrix}$$

$$\textcircled{2} \quad -5x - 4y = (-2+i)y \Rightarrow 5x = (-2-i)y$$

multiply $\textcircled{2}$ by $(-2+i)$

4 pts

$$5(-2+i)x = 5y$$

$$y = (-2+i)x \text{ same as } \textcircled{1}$$

Note $(-2-i)\bar{V} = \begin{pmatrix} -2-i \\ 5 \end{pmatrix}$ is also an eigenvector

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7. Find the solution of the damped harmonic oscillator $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 0$ with initial $y(0) = 4$ and $v(0) = 0$.

a.) (5 pts) Find the general solution $y(t)$ for the equation.

$$\text{try } y = e^{st} \Rightarrow s^2 + 4s + 20 = 0$$

$$s = \frac{-4 \pm \sqrt{16 - 80}}{2} = \frac{-4 \pm \sqrt{-64}}{2} = \frac{-4 \pm 8i}{2}$$

$$s = -2 \pm 4i \quad 2 \text{ pts}$$

$$\begin{aligned} y(t) &= e^{(-2+4i)t} = e^{-2t} e^{(4t)i} = e^{-2t} (\cos(4t) + i \sin(4t)) \\ &= \underbrace{e^{-2t} \cos(4t)}_{y_1(t)} + i \underbrace{e^{-2t} \sin(4t)}_{y_2(t)} \end{aligned}$$

General solution $y(t) = k_1 e^{-2t} \cos(4t) + k_2 e^{-2t} \sin(4t) \quad 3 \text{ pts}$

b.) (2 pts) Write the corresponding equation for the velocity $v(t)$.

$$\begin{aligned} v(t) = \frac{dy}{dt} &= -2k_1 e^{-2t} \cos(4t) - 4k_1 e^{-2t} \sin(4t) - 2k_2 e^{-2t} \sin(4t) + 4k_2 e^{-2t} \cos(4t) \\ v(t) &= k_1 e^{-2t} (-2\cos(4t) - 4\sin(4t)) + k_2 e^{-2t} (-2\sin(4t) + 4\cos(4t)) \end{aligned}$$

c.) (3 pts) Write the solution of the initial value problem.

$$y(0) = 4 = k_1$$

$$\Rightarrow k_1 = 4 \quad k_2 = 2$$

$$v(0) = 0 = -2k_1 + 4k_2$$

$$y(t) = \underbrace{4 e^{-2t} \cos(4t)}_{1 \text{ pt}} + \underbrace{2 e^{-2t} \sin(4t)}_{2 \text{ pts}}$$

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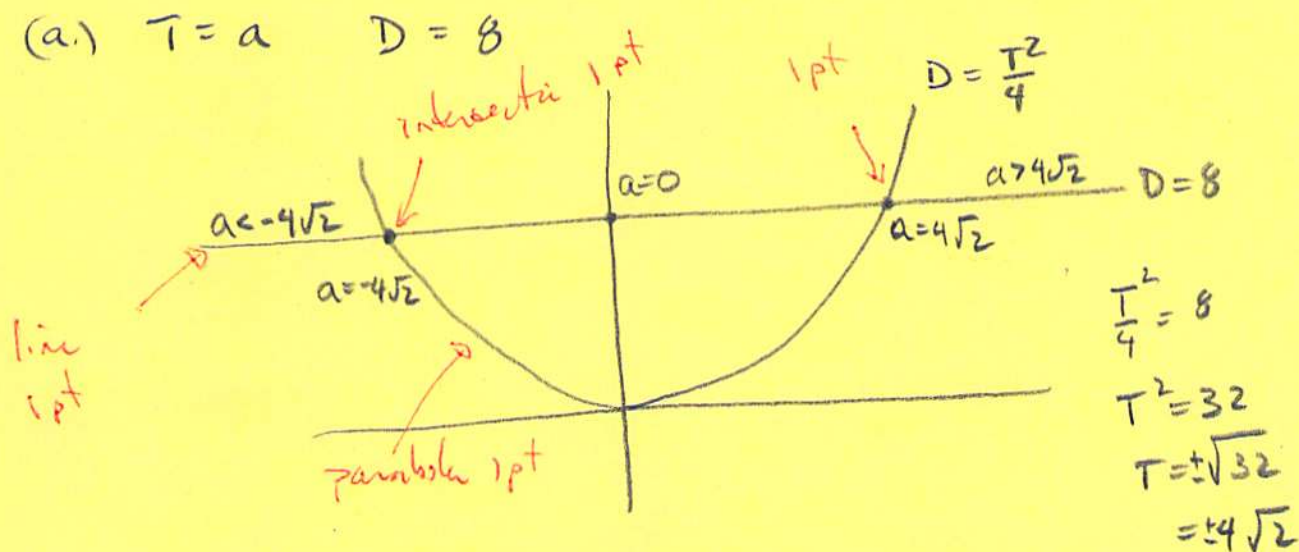
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8. (12 pts) Consider the one parameter family $\frac{d\bar{Y}}{dt} = \begin{pmatrix} a & 4 \\ -2 & 0 \end{pmatrix} \bar{Y}$.

(a) (4 pts) Compute the trace and determinant for the coefficient matrix and draw the corresponding curve in the trace determinant plane.

(b) (6 pts) List the different types of behaviors exhibited by the system as the parameter, a , is varied from $-\infty$ to $+\infty$.

(c) (2 pts) Identify the values of the parameter, a , that are bifurcation values.



b.)

$a < -4\sqrt{2}$	sink
$a = -4\sqrt{2}$	repeated root sink
$-4\sqrt{2} < a < 0$	spiral sink
$a = 0$	center
$0 < a < 4\sqrt{2}$	spiral source
$a > 4\sqrt{2}$	source

1 pt each

c.) $a = -4\sqrt{2}$, 0 , and $4\sqrt{2}$ are the bifurcation values

one point each