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MA 226 Section B – Exam 2a

Question Number	Possible Points	Student Score
1	15	
2a	5	
2b	5	
2c	5	
2d	5	
2e	5	
3	10	
4	8	
5	12	
6	8	
7	14	
8	8	
Total Points	100	

You must show your work to receive full credit

Discussion Sections:

B2: Tuesday 4:30-5:30

B3: Tues : 3:30-4:30

B4: Weds: 9-10

B5: Weds: 10-11

B6: Weds: 4:30-5:30

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1. Short Answer (15 pts)

a) Characterize the behavior of the harmonic oscillator with mass m = 2, spring constant k = 1 and damping coefficient b = 3, as under damped, over damped or critically damped. Show the work that justifies your answer.

$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$ eqn for harmonic oscillator (unforced)

try $y = e^{st}$ $\Rightarrow$ $ms^2 + bs + k = 0$ characteristic polynomial

$s = \frac{-b \pm \sqrt{b^2 - 4 \cdot mk}}{2m} = \frac{-3 \pm \sqrt{9 - 4 \cdot 2 \cdot 1}}{4} = \frac{-3 \pm 1}{4} =$

$s_1 = \frac{-3-1}{4} = -1$ $s_2 = \frac{-3+1}{4} = -\frac{1}{2}$

overdamped $s_1, s_2 < 0$

b) Find all the equilibrium solutions of the system of differential equations

$\frac{dx}{dt} = x(x-y)$ $\Rightarrow$ $x = 0$ or $x = y$ } both must be true

$\frac{dy}{dt} = (x^2 - 4)(y^2 - 9)$ $\Rightarrow$ $x = \pm 2$ or $y = \pm 3$

$x = 0$ and $y = 3$

$x = 0$ and $y = -3$

$x = 2$ and $y = 2$

$x = -2$ and $y = -2$

$y = 3$ and $x = 3$

$y = -3$ and $x = -3$

equilibrium points $(0,3), (0,-3), (2,2), (-2,-2), (3,3), (-3,-3)$

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c) Suppose $\frac{d\vec{Y}}{dt} = A\vec{Y}$ is a linear system with two distinct real valued eigenvalues. One eigenvalue $\lambda_1 = -1$ has a corresponding eigenvector $\vec{V_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. The two functions $\vec{Y_1}(t) = 2e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\vec{Y_2}(t) = 4e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are both straight line solutions and they trace out the same points for $-\infty < t < \infty$. Why is this not a violation of the Uniqueness Theorem and what is the specific relationship between $\vec{Y_1}(t)$ and $\vec{Y_2}(t)$?

- Not a violation of the Uniqueness Theorem because the two solutions are never at the same point at the same time.

- $\bar{Y}_2(t) = \bar{Y}_1(t - t_0)$

$$4e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = 2e^{-(t-t_0)}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$4 = 2e^{-(t-t_0)}$$

$$2e^{-t} = e^{-t}e^{t_0}$$

$$2 = e^{t_0}$$

$$t_0 = \ln 2$$

$$\boxed{\bar{y}_2(t) = \bar{Y}_1(t - \ln 2)}$$

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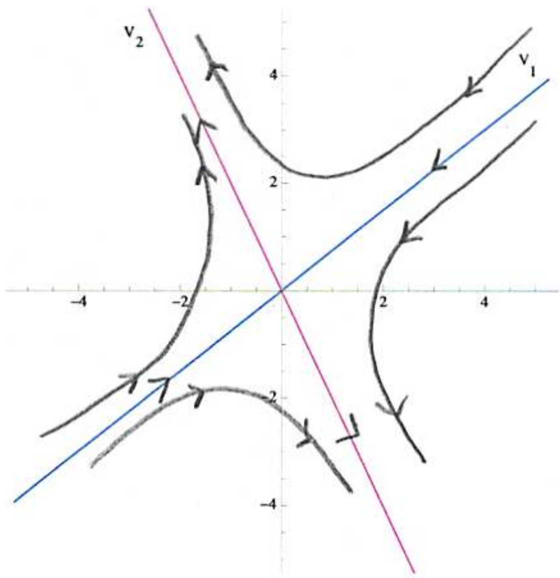
2. (5 pts each)Draw the phase portrait of a linear system given the following information:

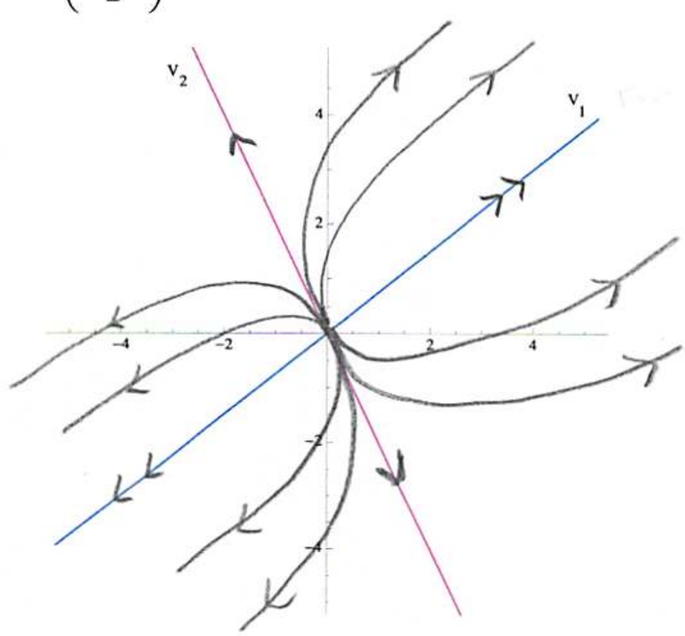
a.) Given eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 1$ with corresponding eigenvectors

$\vec{V_1} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\vec{V_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



b.) Given eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ with corresponding eigenvectors

$\vec{V_1} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\vec{V_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



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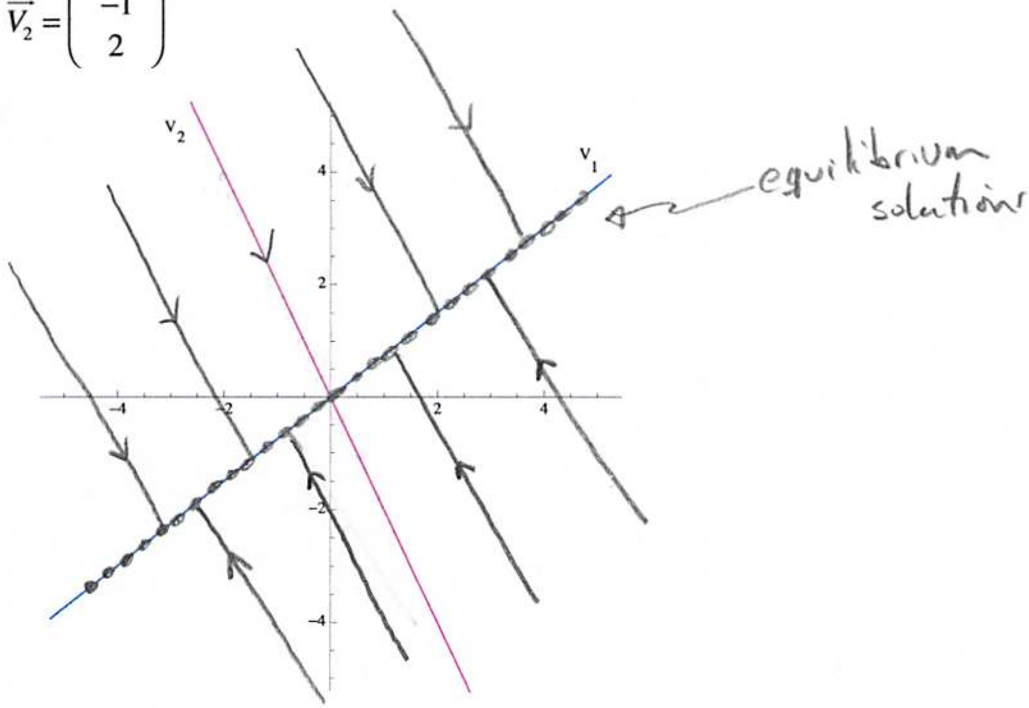
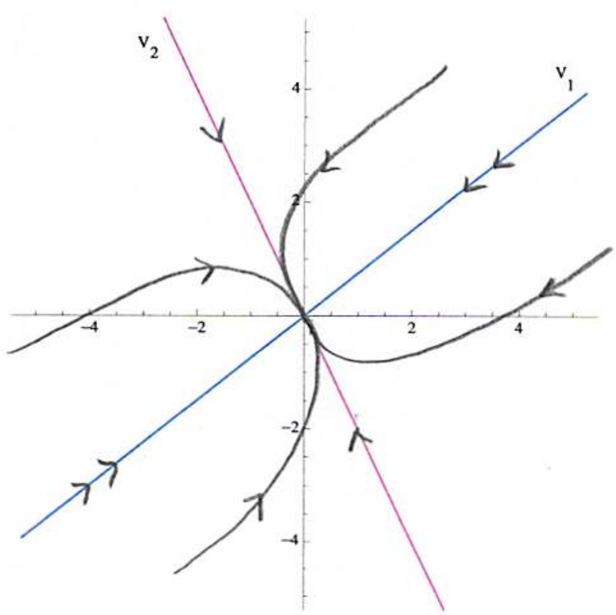
c.) Given eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$ with corresponding eigenvectors

$\vec{V_1} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\vec{V_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



d.) Given eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$ with corresponding eigenvectors

$\vec{V_1} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\vec{V_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



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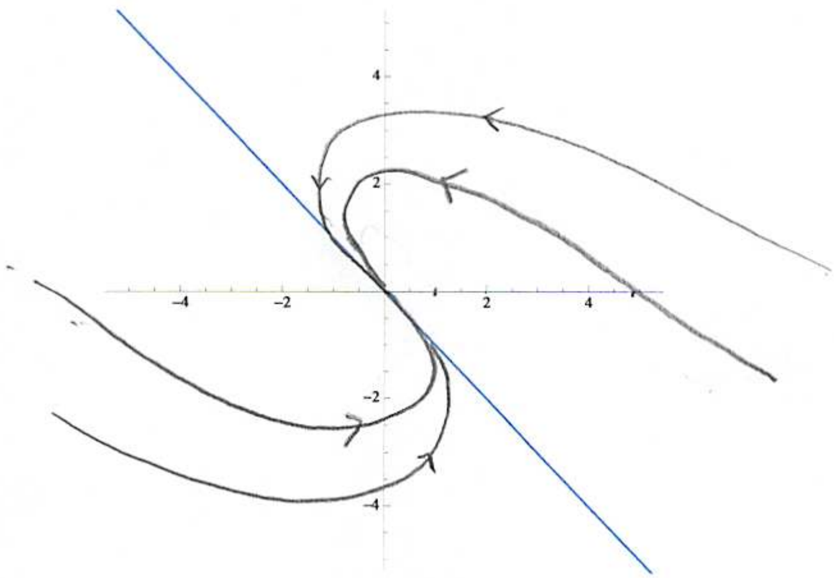
e.) Given the linear system $\frac{d\vec{Y}}{dt} = \begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix}\vec{Y}$ with repeated eigenvalue $\lambda = -1$ and corresponding eigenvector $\vec{V} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ draw the phase portrait. Please show how you determined the correct orientation to use.



To determine orientation choose $(1, 0)$

$$\begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$$ ← Note positive y-component

⇒ moving up and to the left

If we used $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we would expect to find a negative x-component

$$\begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$$

← moving to the let

← moving up

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3. (a) (7 pts) Find the general solution to the partially coupled system **without** using matrix techniques.

$$\frac{dx}{dt} = -3x \quad \Rightarrow \quad x(t) = k_1 e^{-3t}$$

$$\frac{dy}{dt} = 2x - 3y \quad \Rightarrow \quad \frac{dy}{dt} = 2k_1 e^{-3t} - 3y \quad \Rightarrow \quad \frac{dy}{dt} + 3y = 2k_1 e^{-3t} \quad (*)$$

homogeneous solution $\quad \frac{dy}{dt} + 3y = 0 \quad \Rightarrow \quad y_h(t) = k_2 e^{-3t}$

particular solution $\quad$ must use $y_p(t) = Ate^{-3t}$ because $y_h(t)$ has e^{-3t} term

$$y_p'(t) = Ae^{-3t} - 3Ate^{-3t}$$

plug into $(*)$

$$(Ae^{-3t} - 3Ate^{-3t}) + 3(Ate^{-3t}) = 2k_1 e^{-3t}$$

$$Ae^{-3t} = 2k_1 e^{-3t} \qquad \boxed{A = 2k_1}$$

General Solution

$$x(t) = k_1 e^{-3t}$$

$$y(t) = k_2 e^{-3t} + 2k_1 t e^{-3t}$$

(b) Find the solution with the initial value $(x_0, y_0) = (4,3)$

$$x(0) = k_1 = 4$$

$$y(0) = k_2 + 0 = 3$$

$$x(t) = 4e^{-3t}$$

$$y(t) = 3e^{-3t} + 8te^{-3t}$$

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4. (8 pts) Use Euler's Method with a step size of .25 to approximate the solution of the initial value problem at time t = .75.

$$\frac{dx}{dt} = -3x = f(x,y)$$
$$\frac{dy}{dt} = 2x - 3y = g(x,y)$$
where $(x_0, y_0) = (4,3)$

$x_K + \Delta t\, f(x_K, y_K)$ $y_K + \Delta t\, g(x_K, y_K)$

K	t_K	x_K	y_K	$f(x_K, y_K)$	$g(x_K, y_K)$	new x_K	new y_K
0	0	4	3	-12	-1	1	2.75
1	.25	1	2.75	-3	-6.25	.25	1.1875
2	.50	.25	1.1875	-.75	-3.0625	.0625	.421875
3	.75	.0625	.421875				

$x(.75) \approx .0625$

$y(.75) \approx .421875$

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5. (12 pts) Solve the initial value problem: $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{Y}$ with initial condition $\vec{Y}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$. Be sure to show the general solution too. What type of equilibrium point do we have at the origin?

step 1 : eigenvalues $\quad \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 0$

$$(1-\lambda)(1-\lambda)-4 = 0$$

$$\lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda-3)(\lambda+1) = 0$$

eigenvalues

$\lambda_1 = -1 , \lambda_2 = 3$

step 2 eigenvectors

$\lambda_1 = -1$

$x + y = -x$

$4x + y = -y$

$\Rightarrow y = -2x \qquad \bar{V}_1 = \begin{pmatrix} x \\ -2x \end{pmatrix}$ let $x=1$ $\bar{V}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = 3$

$x + y = 3x$

$4x + y = 3y$

$\Rightarrow y = 2x \qquad \bar{V}_2 = \begin{pmatrix} x \\ 2x \end{pmatrix}$ let $x=1$ $\bar{V}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

General Solution

$$\boxed{\bar{y}(t) = k_1 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$\boxed{\lambda_1 < 0 \text{ and } \lambda_2 > 0 \Rightarrow \text{equilibrium point is a saddle}}$

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6. (8 pts) The linear system $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\vec{Y}$ has complex eigenvalues.

a.) Compute the eigenvalues

$$\det\begin{bmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(1-\lambda) + 65 = 0$$

$$\lambda^2 - 4\lambda + 3 + 65 = 0$$

$$\lambda^2 - 4\lambda + 68 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4\cdot 68}}{2}$$

$$= \frac{4 \pm \sqrt{-256}}{2} = \frac{4 \pm 16i}{2}$$

$$\boxed{\lambda = 2+8i \text{ or } 2-8i}$$

b.) Compute one eigenvector $\lambda = 2+8i$

$$3x - 13y = (2+8i)x \Rightarrow -13y = (-1+8i)x \quad ①$$

$$5x + y = (2+8i)y \Rightarrow 5x = (1+8i)y \quad ②$$

mult ① by $(-1-8i)$

$$-(-1-8i)13y = 65x$$

$$x = \frac{13}{65}(1+8i)y$$

$$x = \frac{1}{5}(1+8i)y$$

$$\bar{V} = \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

mult ② by $(1-8i)$

$$5(1-8i)x = 65y$$

$$\frac{5}{65}(1-8i)x = y$$

$$\frac{1}{13}(1-8i)x = y$$

$$\bar{V} = \begin{pmatrix} 13 \\ 1-8i \end{pmatrix}$$

both answers are correct
they are constant multiples of each other.
(complex)

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7. The linear system $\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\vec{Y}$ has complex eigenvalues. One eigenvalue is $\lambda = 4+2i$ and the corresponding eigenvector is $\vec{V} = \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$.

a.) (8 pts) Find two linearly independent real valued functions that solve the system and write the general solution for the system.

$\bar{Y}(t) = e^{\lambda t}\bar{V}$ is a complex solution. Both real and imaginary parts are also solutions.

$$\bar{y} = e^{(4+2i)t}\begin{pmatrix} 1-i \\ 2 \end{pmatrix} = e^{4t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1-i \\ 2 \end{pmatrix}$$

$$\bar{Y}(t) = e^{4t}\begin{pmatrix} \cos 2t - \sin 2t + i(\sin 2t - \cos 2t) \\ 2\cos 2t + i(2\sin 2t) \end{pmatrix}$$

$$\bar{Y}(t) = \underbrace{e^{4t}\begin{pmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{pmatrix}}_{\bar{Y}_1(t)} + i\,\underbrace{e^{4t}\begin{pmatrix} \sin 2t - \cos 2t \\ 2\sin 2t \end{pmatrix}}_{\bar{Y}_2(t)}$$

General solution

$\bar{Y}(t) = k_1\bar{Y}_1 + k_2\bar{Y}_2$

$\bar{Y}_1(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\bar{Y}_2(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

linearly independent

b.) (4 pts.)Find the particular solution that satisfies the initial condition $\vec{Y}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\bar{Y}(0) = k_1\bar{Y}_1(0) + k_2\bar{Y}_2(0) = k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$k_1 - k_2 = -1 \Rightarrow k_2 = 2$

$2k_1 = 2 \Rightarrow k_1 = 1$

$$\boxed{\bar{y}(t) = e^{4t}\begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} + 2e^{4t}\begin{pmatrix} \sin 2t - \cos 2t \\ 2\sin 2t \end{pmatrix}}$$

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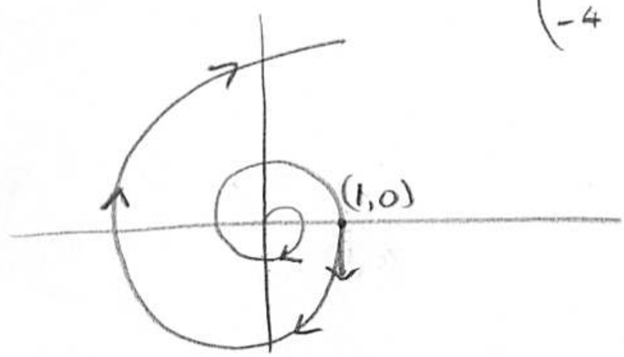
Problem # 7 Continued: $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \vec{Y}$

c.) (2 pts) Classify the type of equilibrium point we find at the origin. What is it's orientation? Clockwise or counter clockwise?

Equilibrium point is a <u>spiral source</u> because the real part of $\lambda > 0$.

$$\begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

negative y implies clockwise spin



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8. Trace determinant plane. Using the definitions of Trace and Determinant we can express the eigenvalues of a 2x2 linear system as $\lambda = \dfrac{T \pm \sqrt{T^2 - 4D}}{2}$

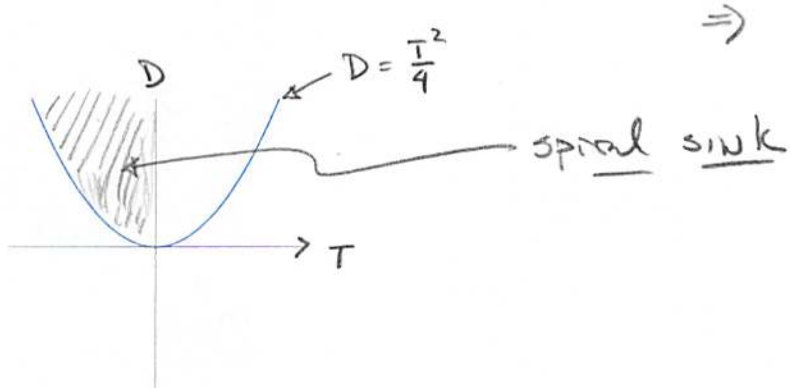
For each of the linear systems given below: (i) Compute the Trace and the Determinant (ii) Shade in the region of the Trace determinant plane where the specified matrix lives. (iii) State the behavior at the equilibrium point of systems in this region.

a.) $\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \vec{Y}$

$T = -4 \qquad D = 4 + 9 = 13$

$T^2 - 4D = 16 - 52 = -36$

$\Rightarrow$



b.) $\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \vec{Y}$

$T = 5 \qquad D = 4$

$T^2 - 4D = 25 - 16 = 9$