Name: Prof D

Discussion Section : ______

MA 226 Section B – Exam 2a

Question Number	Possible Points	Student Score
1	15	
2a	5	
2b	5	
2c	5	
2d	5	
2e	5	
3	10	
4	8	
5	12	
6	8	
7	10	
8	12	
Total Points	100	

You must show your work to receive full credit

Discussion Sections:

B2: Wednesday 9-10

B3: Wednesday 2-3

Name:________________________ Discussion Section : ______

1. Short Answer (15 pts)

a) Characterize the behavior of the harmonic oscillator with mass m = 1, spring constant k = 9 and damping coefficient b = 6, as underdamped, overdamped or critically damped. Show the work that justifies your answer.

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$$

$$s^2 + 6s + 9 = 0$$

$s = -3$ repeated root

critically dambed $\quad 6^2 - 4k = 36 - 4\cdot 9 = 0$

b) Find all the equilibrium solutions of the system of differential equations

$$\frac{dx}{dt} = 3\left(1 - \frac{x}{3}\right)x - xy$$

$$\frac{dy}{dt} = -16y + 4xy$$

Ask

$(3-x)x - xy = 0$

$-16y + 4xy = 0$

$x(3-x-y) = 0$

$y(-16+4x) = 0$

$3-x-y = 0 \Rightarrow 3-4-y=0 \Rightarrow -1 = y$

$-16 + 4x = 0 \Rightarrow x = 4$

Equilibrium points

$x=0, \; y=0$ 1pt

$x=3, \; y=0$ 2pt

$x=4, \; y=-1$ 2pts

Name:_________________________ Discussion Section : _______

c) Consider the two systems of differential equations

(i)

$$\frac{dx}{dt} = 0.4x - 4xy$$

$$\frac{dy}{dt} = -3y + 0.1xy$$

(ii)

$$\frac{dx}{dt} = 0.4x - 0.2xy$$

$$\frac{dy}{dt} = -0.2y + 3xy$$

One of these systems refers to a predator-prey system with very lethargic predators- those who seldom catch prey but who can live for a long time on a single prey (for example, boa constrictors). The other system refers to a very active predator that requires many prey to stay healthy (such as a small cat). The prey in each case is the same. Identify which system is which and justify your answer.

lethargic predators have minimal impact on prey population

$\Rightarrow$ (ii) has lethargic predators. $|-0.2xy| < |-4xy|$

$\Rightarrow$ (i) has active predators

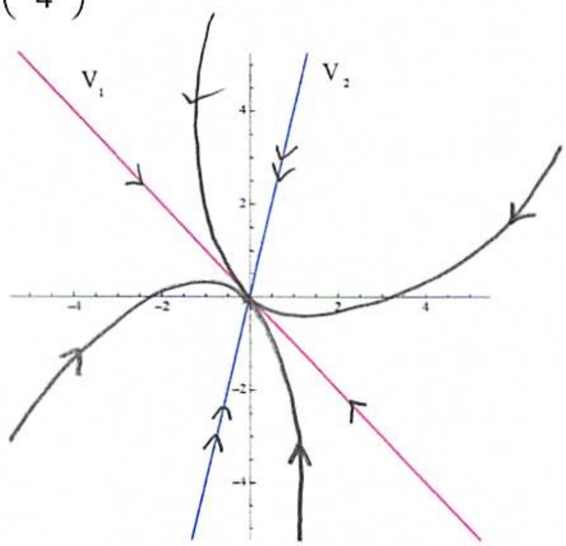
reason is 3 points

Name:_______________________ Discussion Section : ______

2. (5 pts each)Draw the phase portrait of a linear system given the following information:

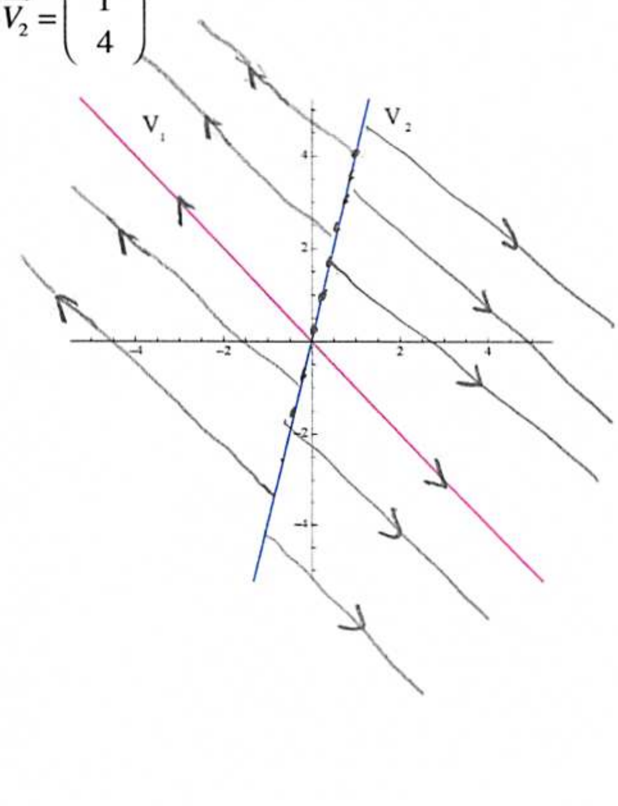
a.) Given eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -5$ with corresponding eigenvectors

$\vec{V_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\vec{V_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$



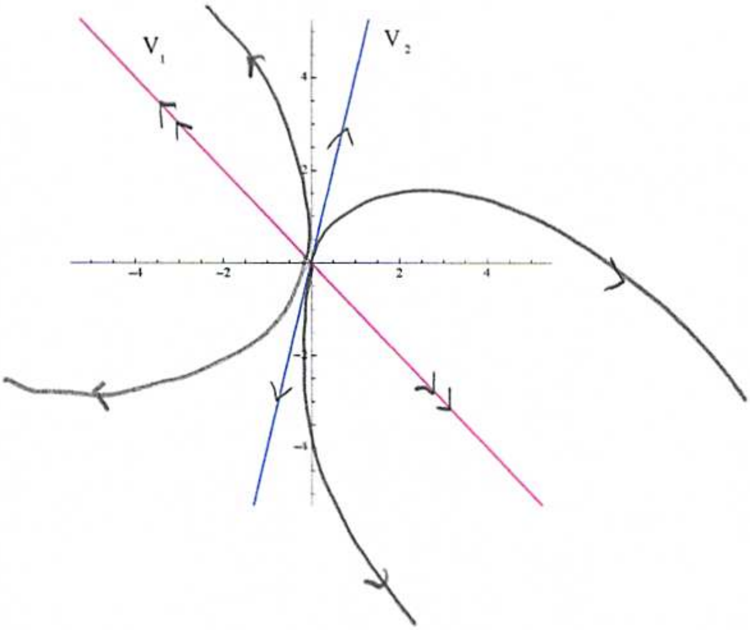
b.) Given eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 0$ with corresponding eigenvectors

$\vec{V_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\vec{V_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

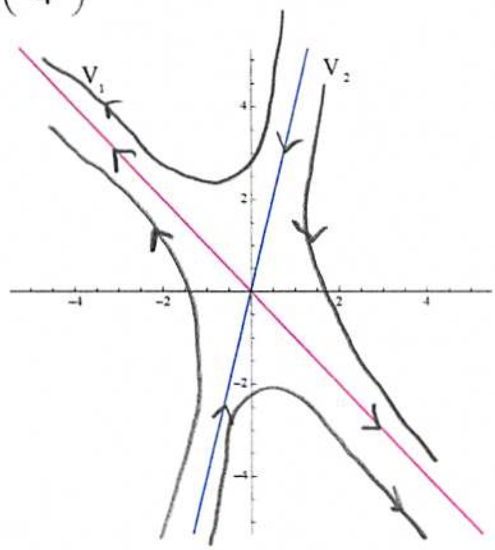


Name:________________________ Discussion Section : ______

c.) Given eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ with corresponding eigenvectors $\vec{V_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\vec{V_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$



d.) Given eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ with corresponding eigenvectors $\vec{V_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\vec{V_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$



Name:______________________ Discussion Section : ______

e.) Draw the phase portrait for the linear system $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\vec{Y}$ with complex eigenvalue $\lambda = 4 + 2i$. Please show how you determined the correct orientation.

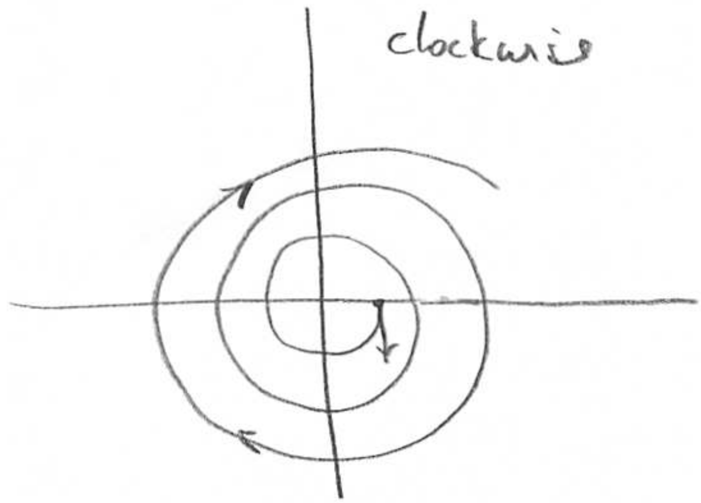
spiral source

-2pts real$(\lambda) > 0$

$\begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ y is negative

3 points

clockwise

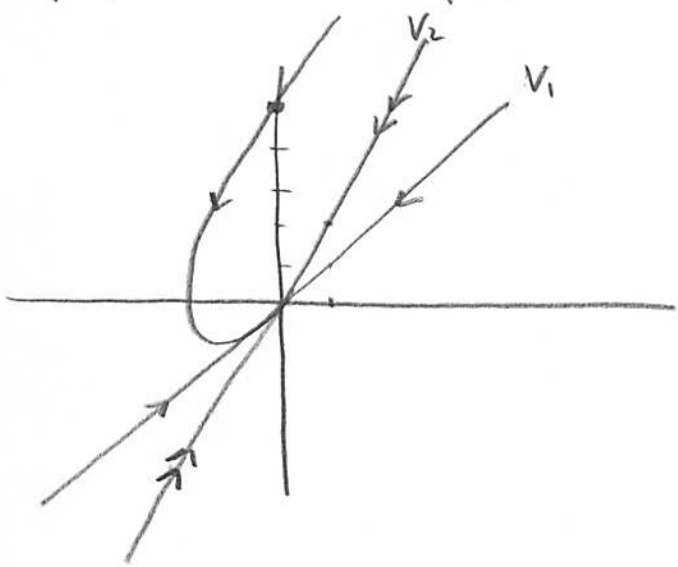


Name:________________________ Discussion Section : ______

3. (a)(5 pts) Given the linear system $\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix}\vec{Y}$ with straight line solutions

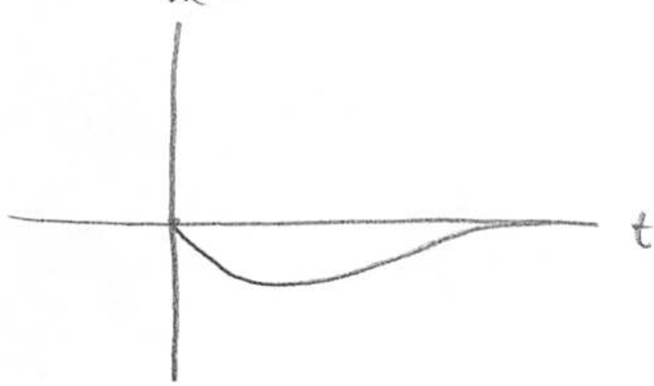
$\vec{Y_1}(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix}$ and $\vec{Y_2}(t) = \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix}$. Make a rough sketch of the solution of the initial value problem passing through the point $(0,5)$.

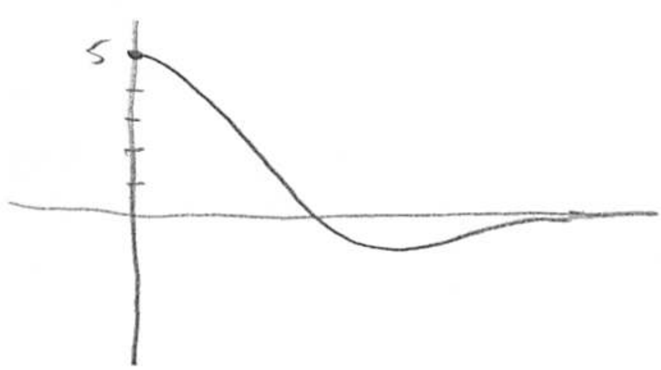
$\bar{Y}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\bar{Y}_2(t) = e^{-4t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$



(b) (5 pts) Draw the corresponding $x(t)$ and $y(t)$ graphs

$x(t)$





Name:_______________________   Discussion Section : _______

4. (8 pts) Use Euler's Method with a step size of .25 to approximate the solution of the initial value problem at time t = .75. Use 6 decimal places of accuracy.

$$\frac{dx}{dt} = y \quad = f(x,y)$$
$$\frac{dy}{dt} = 2x - y \quad = g(x,y)$$

where at t =0, we have $(x_0, y_0) = (1,2)$

k	t_k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	x_{k+1}	y_{k+1}
0	0	1	2	2	0	1.5	2
1	.25	1.5	2	2	1	2	2.25
2	.5	2	2.25	2.25	1.75	2.5625	2.6875
3	.75	2.5625	2.6875				

1 point each

Name:__________________ Discussion Section : ______

5. Solve the initial value problem: $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\vec{Y}$ with initial condition $\vec{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

a. (7 pts) Write the general solution.

b. (2 pts)Write the solution of the initial value problem.

c. (3 pts) Make a rough sketch of the solution in the phase plane.

Eigenvalues

$$\det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(4-\lambda) + 1 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda-3)(\lambda-3) = 0$$

$\lambda = 3$ repeated   2 pts

Eigenvector

$$\left.\begin{aligned} 2x + y &= 3x \\ -x + 4y &= 3y \end{aligned}\right\} \quad \left.\begin{aligned} -x+y &= 0 \\ -x+y &= 0 \end{aligned}\right\} \Rightarrow y = x$$

$$\bar{V} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

General Solution   2 pts

a) $$\boxed{\bar{Y}(t) = e^{3t}\bar{V}_0 + te^{3t}\bar{V}_1}$$

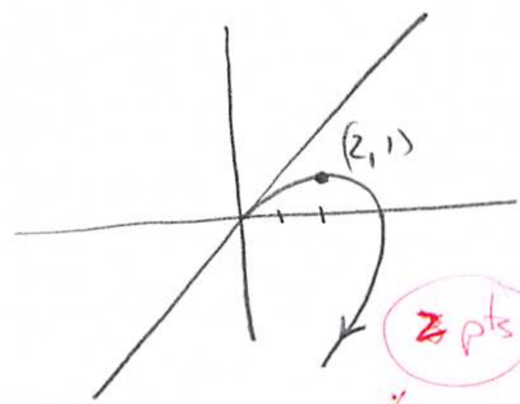
$$V_1 = (A - 3I)\bar{V}_0 \qquad \text{3 pts}$$

$$= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}$$

Solution of Initial Value Problem

b) $$\bar{Y}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + te^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

c)



2 pts

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$ y is negative

Name:________________________ Discussion Section : ______

6. (8 pts) The linear system $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \vec{Y}$ has complex eigenvalues.

a.) Compute the eigenvalues

4pts

$$\det \begin{pmatrix} -\lambda & 1 \\ -5 & -4-\lambda \end{pmatrix} = -\lambda(-4-\lambda) + 5 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 - 4 \cdot 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$\lambda = -2 \pm i$$

b.) Compute one eigenvector $A\bar{v} = \lambda \bar{v}$

$$0x + y = (-2+i)x \Rightarrow \boxed{y = (-2+i)x} \quad ①$$

$$-5x - 4y = (-2+i)y \Rightarrow -5x = (2+i)y \Rightarrow 5x = (-2-i)y \quad ②$$

mult. ② by $(-2+i)$ ② $\Rightarrow$ $5(-2+i)x = 5y$

$$\boxed{y = (-2+i)x}$$

4pts

$$\boxed{\bar{v} = \begin{pmatrix} 1 \\ -2+i \end{pmatrix}}$$

also $(-2-i)\bar{v} = \boxed{\begin{pmatrix} -2-i \\ 5 \end{pmatrix}}$ is also an eigenvector

Name:________________________ Discussion Section : ______

7. Find the solution of the damped harmonic oscillator $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+20y=0$ with initial $y(0)=4$ and $v(0)=0$.

a.) (5 pts) Find the general solution $y(t)$ for the equation.

b.) (2 pts) Write the corresponding equation for the velocity $v(t)$.

c.) (3 pts) Write the solution of the initial value problem.

a.) $y(t)=e^{st}$ $\quad s^2+4s+20=0$

$$s=\frac{-4\pm\sqrt{16-80}}{2}=\frac{-4\pm 8i}{2}=-2\pm 4i$$ 2pts

$$y(t)=e^{-2t}e^{(4t)i}=e^{-2t}(\cos(4t)+i\sin(4t))$$

$$=\underbrace{e^{-2t}\cos(4t)}_{y_1(t)}+i\underbrace{e^{-2t}\sin(4t)}_{y_2(t)}$$

$$\boxed{y(t)=k_1e^{-2t}\cos(4t)+k_2e^{-2t}\sin(4t)}$$ 3pts

2pts b.) $\vec{v}(t)=k_1e^{-2t}(-2\cos(4t)-4\sin(4t))+k_2e^{-2t}(-2\sin(4t)+4\cos(4t))$

c.) $y(0)=4 = k_1$

$v(0)=0 = -2k_1+4k_2$

$k_1=4$

$k_2=\frac{1}{2}k_1=2$

$$\boxed{y(t)=4e^{-2t}\cos(4t)+2e^{-2t}\sin(4t)}$$

1pt 2pts

Name:________________________ Discussion Section : ______

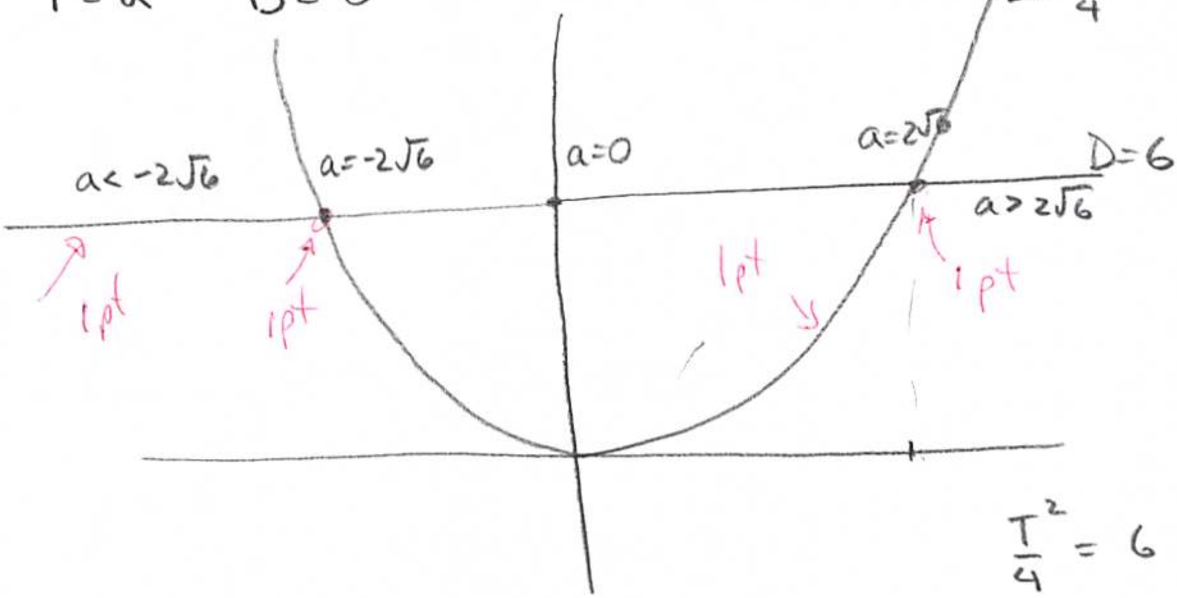
8. (12 pts) Consider the one parameter family $\frac{d\vec{Y}}{dt} = \begin{pmatrix} a & 3 \\ -2 & 0 \end{pmatrix}\vec{Y}$.

(a) (4 pts)Compute the trace and determinant for the coefficient matrix and draw the corresponding curve in the trace determinant plane.

(b)(6pts) List the different types of behaviors exhibited by the system as the parameter , a, is varied from $-\infty$ to $+\infty$.

(c)(2 pts) Identify the values of the parameter , a , that are bifurcation values.

a) $T = a$ $D = 6$



$\frac{T^2}{4} = 6$

$T^2 = 24$

$T = \pm 2\sqrt{6}$

b)

1 pt each

$a < -2\sqrt{6}$	sink
$a = -2\sqrt{6}$	repeated root sink
$-2\sqrt{6} < a < 0$	spiral sink
$a = 0$	center
$0 < a < 2\sqrt{6}$	spiral source
$a = 2\sqrt{6}$	repeated root source
$a > 2\sqrt{6}$	source

c) $a = -2\sqrt{6},\ 0,\ 2\sqrt{6}$ are bifurcation values

one point each