Prof. Deutsch Fall 2014

MA226 Section B – Exam 2 Study Guide

This exam will cover the material presented in Chapter 2 & 3. Questions will be based primarily on the homework problem and also the examples discussed in lecture which can all be found in the text. You will be permitted the use of a calculator on the exam but no cell phones, ipads or laptops can be used.

Sec 2.1 – Modeling via Systems

- Homework problems: 1,7,15,19,20,21,22
- All of the homework problems are relevant to what I would ask on this exam.
- Presentation of a variety of predator prey models. Students should be able to distinguish between models where the predators or prey are slow or fast or when they are large or small. See problems 1&15
- Derivation of the undamped harmonic oscillator from Hooke's Law. Students should feel comfortable with trigonometric solutions of the corresponding 2^{nd} order differential equations and the associated natural frequency. See problem 20.
- The representation of the 2^{nd} order differential equation as a linear system as shown on Page 159

Sec 2.2 – The Geometry of Systems

- Homework problems: 1,3,5,7,9,12,13
- All of the homework problems are relevant to what I would ask on this exam
- Given a linear system students should be able to sketch the direction field for a limited number of points and to match a system to a direction field
- Find equilibrium solutions of linear systems as in questions 12 & 13

Sec 2.3 – The Damped Harmonic Oscillator

- Homework problems: 1,3,5,7
- All homework problems are relevant to what I would ask on this exam
- Students will be expected to write the equation for the damped harmonic oscillator given values of the parameters : m, b, and k
- Students will be expected to use the "guessing method" shown on P.185 to find solutions of the damped harmonic oscillator.
- Using the "guessing method" to generate the corresponding straight line solutions of the damped harmonic oscillator in the phase plane.

Sec 2.4 – Additional Analytic Methods for Special Systems

- Homework problems: 1, 3, 5, 7, 9
- All homework problems are relevant to what I would ask on this exam
- Check if a vector valued function is a solution of a system. (HW 1, 3)
- Find a solution of a partially coupled system using the techniques from chapter 1.

Sec 2.5 – Euler's Method for Systems

- Homework problems: 1, 3, 5
- Only homework problem 1 is relevant to what I will ask on the exam
- Given a linear system and an initial condition and a step size, use Euler's Method to approximate the solution 3 time steps away.

Sec 2.6 – Existence and Uniqueness for Systems

- Homework problems: 1, 3, 4, 5, 8, 9
- All homework problems are relevant to what I would ask on this exam
- The Existence Theorem guarantees that the solution exists for some interval around time t_0.
- The Uniqueness Theorem says that no two solutions of the system can be at the same point at the same time.
- When drawing solutions in the phase plane it appears that two periodic solutions can sweep out the same curve but this is not a violation of the Uniqueness Theorem. Problems 5,8, and 9 address this fact.

Sec 3.1 Properties of Linear Systems and the Linearity Principle

- Homework problems: 5,6,8,9,25,26,27,28,33,34
- All homework problems are relevant to what I would ask on this exam
- Linear systems have the form : $\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

  where a,b,c and d are all constant real numbers.
- The origin is always an equilibrium point for every linear system
- Only when the det A = 0 will we have an opportunity for equilibrium points in addition to the origin which we will see in section 3.5.
- Linearity Principle P. 249 . Two key facts: (i) If $\vec{Y}(t)$ is a solution of a linear system then $k{\cdot}\vec{Y}(t)$ is also a solution of the system (ii) If $\vec{Y_1}(t)$ and $\vec{Y_2}(t)$ are two solutions of a linear system then their sum $\vec{Y_1}(t)+\vec{Y_2}(t)$ is also a solution.

- General Solution of Linear Systems Theorem P.256

- Given two linearly independent solutions of a linear system students need to be able to construct the general solution and use it to solve an initial value problem (See problems 25,26,27, 28)

Sec 3.2 Straight Line Solutions

- Homework problems: 1,3,5,11,19,20,21
- All homework problems are relevant to what I would ask on the exam.
- Two straight line solutions are found when the characteristic equation yields two distinct real valued eigenvalues. Each eigenvalue has a corresponding eigenvector and these two eigenvectors will be linearly independent. The general solution in this case will be:
  $\vec{Y}(t) = k_1 e^{\lambda_1 t}\vec{V_1} + k_2 e^{\lambda_2 t}\vec{V_2}$
- Given a general solution you need to be able to find $k_1$and k_2 to satisfy any initial condition.

Sec 3.3 – Phase Portraits for Linear Systems with Real Eigenvalues

- Homework Problems : 1,3,5,9,13,19,21
- All homework problems are relevant to what I would ask on the exam.
- When linear systems have two distinct real eigenvalues there are three possible phase portraits. The equilibrium point at the origin can either be a sink, a saddle, or a source. Students will need to know how to sketch the graphs of all three cases.
- Saddles and sources are said to be unstable equilibrium points because a small disturbance from equilibrium can put a point on a curve that leads away from the origin.
- Sinks are said to be stable equilibrium points because a small disturbance from equilibrium will put the point on a curve leading back to the origin.
- The summary on page 290 of the three types of equilibrium points is very important in how phase portraits are drawn for straight line solutions.

Sec 3.4 – Complex Eigenvalues

- Homework Problems: 1,3,5,7,9,11
- All homework problems are relevant to what I would ask on the exam.
- When the eigenvalues found in the characteristic polynomial are complex numbers then the solutions will either be in an elliptical orbit about the origin, or spiral into the origin, or spiral away from the origin.
- Euler's Formula : $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
- The key to finding two linearly independent solutions in the case of complex eigenvalues is the fact that when a complex function solves the linear system then both the real part of the function and the imaginary part of the function both solve the system. These are both real valued functions that are also linearly independent and we use them to generate the general solution.

- If the real part of the complex eigenvalue is negative then the solutions spiral toward the origin and the origin is called a spiral sink.
- If the real part of the complex eigenvalue is positive then the solutions spiral away from the origin and the origin is called a spiral source
- If the real part of the eigenvalue is zero then the origin is called a center and the solutions live on either a circle or an ellipse with the major axis centered on the origin.

Sec 3.5 – Special Cases and Zero Eigenvalues

- Homework Problems: 1,3,5,7,9,17,19,21,22,23
- All homework problems are relevant to what I would ask on the exam.
- Repeated non zero real eigenvalues imply that the phase line only has one direction of straight line solutions.
- The general solution for the repeated eigenvalue λ is given on page 319 as : $\vec{Y}(t) = e^{\lambda t}\vec{V_0} + te^{\lambda t}\vec{V_1}$ where $\vec{V_1} = (A - \lambda I)\vec{V_0}$.
- Systems with zero as an eigenvalue will always have a line of equilibrium points in the direction of the eigenvector corresponding to the zero eigenvalue. The general solution has the form $\vec{Y}(t) = k_1\vec{V_1} + k_2 e^{\lambda_2 t}\vec{V_2}$ where $\lambda_1 = 0$.

Sec 3.6 – Second Order Linear Equations

- Homework Problems: 1,3,7,9,13,15,21,23
- All homework problems are relevant to what I would ask on the exam.
- Much of this section is review of the second order differential equation for the damped harmonic oscillator as shown in section 2.3 . However, in the special case where the damped harmonic oscillator has complex eigenvalues leading to complex solutions we are shown in this section a method that is significantly easier when finding the general solution then the way we needed to proceed in section 3.4. See problems : 1, 3, 5, 7, 9, 11 .
- The terms underdamped, overdamped, and critically damped describe all possible solutions of the damped harmonic oscillator. Each type of behavior can be determined from the parameters m, b, and k as shown on page 337.

Sec 3.7 – The Trace Determinant Plane

- Homework Problems: 2,4,6,11
- All homework problems are relevant to what I would ask on the exam.
- Being able to identify the bifurcation values in the one parameter families like questions 2, 4, 6, and 11 as well as describe the different types of behavior of the system as the parameter is varied is the type of question that you should expect to see from this section.
- Students should be familiar with the expression for the characteristic polynomial written in terms of the trace ,T, and the determinant, D, as shown on page348.

- Students should understand Figure 3.48 on page 351 which classifies all linear systems in terms of only the Trace and the Determinant.