Sec 6.4 Homework

1.) Compute the limit

$$\lim_{\Delta t \to 0} \left(\frac{e^{s\Delta t} - e^{-s\Delta t}}{2\Delta t} \right) \qquad \frac{0}{0}$$

Apply L'Hopital's Rule

differentiate w.r.t. Δt

$$\lim_{\Delta t \to 0} \frac{s e^{s\Delta t} + s e^{-s\Delta t}}{2}$$

$$\lim_{\Delta t \to 0} s\left(\frac{e^{s\Delta t} + e^{-s\Delta t}}{2} \right)$$

$$s \cdot \frac{e^{0} + e^{0}}{2} = s \cdot \frac{2}{2} = s$$

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3.) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \delta_3(t) \qquad y(0)=1 \ , \ y'(0)=1$

$$s^2 \mathcal{L}[y] - s\,y(0) - y'(0) + 2\left[s\mathcal{L}[y] - y(0)\right] + 5\mathcal{L}[y] = e^{-3s}$$

$$(s^2+2s+5)\mathcal{L}[y] - s - 1 - 2 = e^{-3s}$$

$$\mathcal{L}[y] = \frac{s+3}{s^2+2s+5} + \frac{e^{-3s}}{s^2+2s+5}$$

$$= \frac{s+3}{(s+1)^2+2^2} + \frac{e^{-3s}}{(s+1)^2+2^2}$$

$$\mathcal{L}[y] = \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2} + \frac{1}{2}\cdot\frac{2\cdot e^{-3s}}{(s+1)^2+2^2}$$

$$y(t) = e^{-t}\cos(2t) + e^{-t}\sin(2t) + \frac{1}{2}u_3(t)e^{-(t-3)}\sin(2(t-3))$$

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4.) $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = -2\delta_2(t) \qquad y(0)=2, \; y'(0)=0$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 2\mathcal{L}\left[\frac{dy}{dt}\right] + 2\mathcal{L}[y] = \mathcal{L}[-2\delta_2(t)]$$

$$s^2\mathcal{L}[y] - s\,y(0) - y'(0) + 2[s\mathcal{L}[y] - y(0)] + 2\mathcal{L}[y] = -2e^{-2s}$$

$$(s^2+2s+2)\mathcal{L}[y] - 2s - 2 = -2e^{-2s}$$

$$\mathcal{L}[y] = \frac{2(s+1)}{s^2+2s+2} - \frac{2e^{-2s}}{s^2+2s+2}$$

$$\mathcal{L}[y] = \frac{2(s+1)}{(s+1)^2+1} - \frac{2e^{-2s}}{(s+1)^2+1}$$

$$y(t) = 2e^{-t}\cos(t) - 2u_2(t)e^{-(t-2)}\sin(t-2)$$

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5.) $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = \delta_1(t) - 3\delta_4(t) \qquad y(0) = y'(0) = 0$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 2\mathcal{L}\left[\frac{dy}{dt}\right] + 3\mathcal{L}[y] = \mathcal{L}[\delta_1(t)] - \mathcal{L}[\delta_4(t)]$$

$$s^2\mathcal{L}[y] - s\,y(0) - y'(0) + 2[s\mathcal{L}[y] - y(0)] + 3\mathcal{L}[y] = e^{-s} - 3e^{-4s}$$

$$(s^2 + 2s + 3)\mathcal{L}[y] = e^{-s} - 3e^{-4s}$$

$$\mathcal{L}[y] = \frac{e^{-s}}{(s^2+2s+1)+2} - \frac{3e^{-4s}}{(s^2+2s+1)+2}$$

$$\mathcal{L}[y] = \frac{1}{\sqrt{2}}\frac{\sqrt{2}e^{-s}}{(s+1)^2+2} - \frac{3}{\sqrt{2}}\frac{\sqrt{2}e^{-4s}}{(s+1)^2+2}$$

$$y(t) = \frac{1}{\sqrt{2}}u_1(t)e^{-(t-1)}\sin\sqrt{2}(t-1) - \frac{3}{\sqrt{2}}u_4(t)e^{-(t-4)}\sin\sqrt{2}(t-4)$$

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6.) (a.) Qualitative behavior of the solution of the IVP

$$\frac{d^2y}{dt^2} + \frac{2dy}{dt} + 3y = \delta_4(t) \qquad y(0)=1,\ y'(0)=0$$

The characteristic polynomial of the unforced oscillator is $\lambda^2+2\lambda+3$, and the eigenvalues are $\lambda = -1 \pm \sqrt{2}i$. Hence, the natural period is $\sqrt{2}\pi$ and the damping causes the solutions of the unforced equation to tend to zero like e^{-t} as $t \to \infty$. At $t=4$, the system is given a jolt, so the solution rises. After $t=4$, the equation is unforced, so the solution again tends to zero as e^{-t}.

b.) $$s^2\mathcal{L}[y] - s\,y(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 3\mathcal{L}[y] = e^{-4s}$$

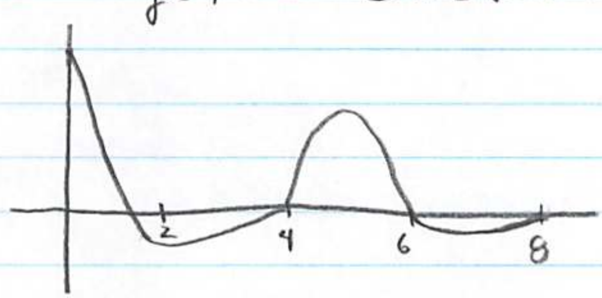
$$(s^2+2s+3)\mathcal{L}[y] - s - 2 = e^{-4s}$$

$$\mathcal{L}[y] = \frac{s+2}{s^2+2s+3} + \frac{e^{-4s}}{s^2+2s+3} =$$

$$\mathcal{L}[y] = \frac{(s+1)+1}{(s+1)^2+2} + \frac{e^{-4s}}{(s+1)^2+2}$$

$$y(t) = e^{-t}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t + \frac{1}{\sqrt{2}}u_4(t)\,e^{-(t-4)}\sin(\sqrt{2}(t-4))$$

c.)



The solution goes through about $\frac{3}{4}$ of a natural period before the application of the delta function. The delta function causes the 2^{nd} maximum of the function to be higher than it would be without the δ function, long term effect is small because damping is large.