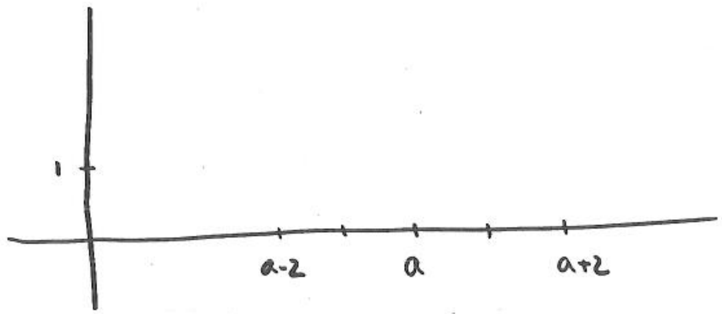
Sec 6.4 – Dirac Delta Function

We are going to define an impulse function. That is a function this is non zero at only one point in its domain.

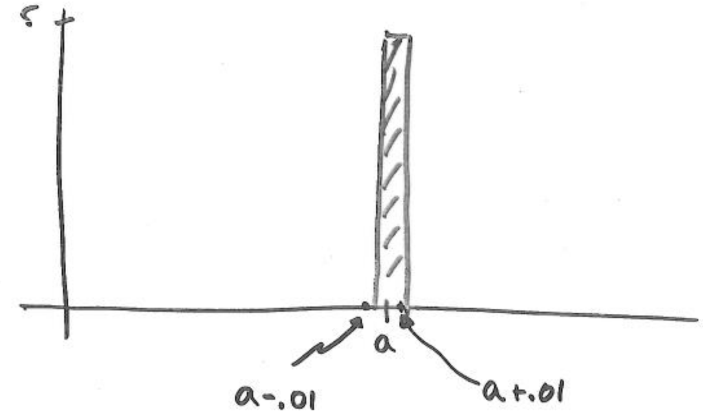
Define: $g_{a,\Delta t}(t) = \begin{cases} \frac{1}{2\Delta t}, & a-\Delta t < t \le a+\Delta t \\ 0 & \text{otherwise} \end{cases}$



$\int_{-\infty}^{\infty} g_{a,2}(t)\,dt =$ $\qquad g_{a,2}(t) = \begin{cases} \frac{1}{4}, & a-2 \le t \le a+2 \\ 0 & \text{otherwise} \end{cases}$

$\int_{-\infty}^{\infty} g_{a,1}(t)\,dt =$ $\qquad g_{a,1}(t) = \begin{cases} \frac{1}{2}, & a-1 \le t \le a+1 \\ 0 & \text{otherwise} \end{cases}$

$\int_{-\infty}^{\infty} g_{a,.1}(t)\,dt =$ $\qquad g_{a,.1}(t) = \begin{cases} 5, & a-.1 \le t \le a+.1 \\ 0 & \text{otherwise} \end{cases}$



$g_{a,.01}(t) = \begin{cases} 50, & a-.01 \le t \le a+.01 \\ 0, & \text{otherwise} \end{cases}$

$\int_{-\infty}^{\infty} g_{a,.01}(t)\,dt = \underline{1}$

$$\boxed{\lim_{\Delta t \to 0} g_{a,\Delta t}(t) = \delta_a(t) \quad \text{Dirac Delta Function}}$$

Recall the Mean Value Theorem for Integrals

$$\int_a^b f(x)\,dx = f(c)(b-a) \quad \text{for some } c \in [a,b]$$

Consider

$$\int_{-\infty}^{\infty} \delta_a(t) f(t)\,dt = \lim_{\Delta t \to 0} \int_{a-\Delta t}^{a+\Delta t} \frac{1}{2\Delta t} f(t)\,dt$$

$$= \lim_{\Delta t \to 0} \frac{1}{2\Delta t} \int_{a-\Delta t}^{a+\Delta t} f(t)\,dt$$

$$= \lim_{\Delta t \to 0} \frac{1}{2\Delta t} f(c) \cdot (a+\Delta t - (a-\Delta t))$$

$$c \in [a-\Delta t, a+\Delta t]$$

$$= \lim_{\Delta t \to 0} \frac{1}{2\Delta t} f(c) \cdot 2\Delta t$$

$$= \lim_{\Delta t \to 0} f(c) \qquad c \in [a-\Delta t, a+\Delta t]$$

$$= f(a)$$

Properties of Dirac Delta Function $\delta_a(t)$

(i) $\delta_a(t) = 0$ for all $t \neq a$

(ii) $\int_{-\infty}^{\infty} \delta_a(t)\,dt = 1$

(iii) $\int_{-\infty}^{\infty} \delta_a(t) f(t)\,dt = f(a)$

Laplace Transform of $\delta_a(t)$

$$\mathcal{L}[\delta_a(t)] = \int_0^{\infty} \delta_a(t) e^{-st}\,dt = e^{-as}$$

$$\mathcal{L}[\delta_a(t)] = e^{-as}$$

(5

Sec 6.4 #3

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \delta_3(t) \quad , \quad y(0)=1,\ y'(0)=1$$

$$s^2\mathcal{L}[y] - s\,y(0) - y'(0) + 2[s\mathcal{L}[y] - y(0)] + 5\mathcal{L}[y] = e^{-3s}$$

$$s^2\mathcal{L}[y] - s - 1 + 2s\mathcal{L}[y] - 2\cdot 1 + 5\mathcal{L}[y] = e^{-3s}$$

$$(s^2+2s+5)\mathcal{L}[y] - s - 1 - 2 = e^{-3s}$$

$$(s^2+2s+5)\mathcal{L}[y] = s+3 + e^{-3s}$$

$$\mathcal{L}[y] = \frac{s+3}{s^2+2s+5} + \frac{e^{-3s}}{s^2+2s+5}$$

$$\mathcal{L}[y] = \frac{s+3}{(s+1)^2+4} + \frac{e^{-3s}}{(s+1)^2+4}$$

$$= \frac{s+1+2}{(s+1)^2+2^2} + \frac{1}{2}\frac{2e^{-3s}}{(s+1)^2+2^2}$$

$$= \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2} + \frac{1}{2}\frac{2e^{-3s}}{(s+1)^2+2^2}$$

$$y(t) = e^{-t}\cos 2t + e^{-t}\sin 2t + \tfrac{1}{2}u_3(t)\,e^{-(t-3)}\sin(2(t-3))$$

Sec 6.3 #33

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 9y = 20\,u_2(t)\sin(t-2) \qquad y(0)=1,\ y'(0)=2$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 4\mathcal{L}\left[\frac{dy}{dt}\right] + 9\mathcal{L}[y] = 20\mathcal{L}\left[u_2(t)\sin(t-2)\right]$$

$$s^2\mathcal{L}[y] - s\,y(0) - y'(0) + 4[s\mathcal{L}[y] - y(0)] + 9\mathcal{L}[y] = \frac{20e^{-2s}}{s^2+1}$$

$$s^2\mathcal{L}[y] - s - 2 + 4s\mathcal{L}[y] - 4 + 9\mathcal{L}[y] = \frac{20e^{-2s}}{s^2+1}$$

$$(s^2+4s+9)\mathcal{L}[y] - s - 6 = \frac{20e^{-2s}}{s^2+1}$$

$$s^2+4s+9\ \mathcal{L}[y] = s+6 + \frac{20e^{-2s}}{s^2+1}$$

$$\mathcal{L}[y] = \frac{s+6}{s^2+4s+9} + \frac{20e^{-2s}}{(s^2+4s+9)(s^2+1)}$$

$$\mathcal{L}[y] = \frac{s+2+4}{(s+2)^2+5} + \frac{(-s+2)e^{-2s}}{s^2+1} + \frac{(s+2)e^{-2s}}{(s+2)^2+5}$$

$$\mathcal{L}[y] = \frac{s+2}{(s+2)^2+5} + \frac{1}{\sqrt{5}}\frac{4\cdot\sqrt{5}}{(s+2)^2+5} - \frac{se^{-2s}}{s^2+1} + \frac{2e^{-2s}}{s^2+1} + \frac{(s+2)e^{-2s}}{(s+2)^2+5}$$

$$y(t) = e^{-2t}\cos\sqrt{5}t + \frac{4}{\sqrt{5}}e^{-2t}\sin\sqrt{5}t$$

Partial Fractions

$$\frac{20}{(s^2+4s+9)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+9}$$

we find $A=-1,\ B=2,\ C=1,\ D=2$

$$\frac{20}{(s^2+4s+9)(s^2+1)} = \frac{-s+2}{s^2+1} + \frac{s+2}{s^2+4s+9}$$

Table 6.1
Frequently Encountered Laplace Transforms.

$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$	$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a} \quad (s > a)$	$y(t) = t^n$	$Y(s) = \dfrac{n!}{s^{n+1}} \quad (s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \dfrac{\omega}{s^2+\omega^2}$	$y(t) = \cos \omega t$	$Y(s) = \dfrac{s}{s^2+\omega^2}$
$y(t) = e^{at} \sin \omega t$	$Y(s) = \dfrac{\omega}{(s-a)^2+\omega^2}$	$y(t) = e^{at} \cos \omega t$	$Y(s) = \dfrac{s-a}{(s-a)^2+\omega^2}$
$y(t) = t \sin \omega t$	$Y(s) = \dfrac{2\omega s}{(s^2+\omega^2)^2}$	$y(t) = t \cos \omega t$	$Y(s) = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s} \quad (s > 0)$	$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$

Table 6.2
Rules for Laplace Transforms:
Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants α and a.

Rule for Laplace Transform	Rule for Inverse Laplace Transform
$\mathcal{L}\left[\dfrac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) = sY(s) - y(0)$	
$\mathcal{L}[y+w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$	$\mathcal{L}^{-1}[Y+W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$
$\mathcal{L}[\alpha y] = \alpha \mathcal{L}[y] = \alpha Y(s)$	$\mathcal{L}^{-1}[\alpha Y] = \alpha \mathcal{L}^{-1}[Y] = \alpha y(t)$
$\mathcal{L}[u_a(t) y(t-a)] = e^{-as} \mathcal{L}[y] = e^{-as} Y(s)$	$\mathcal{L}^{-1}[e^{-as} Y] = u_a(t) y(t-a)$
$\mathcal{L}[e^{at} y(t)] = Y(s-a)$	$\mathcal{L}^{-1}[Y(s-a)] = e^{at} \mathcal{L}^{-1}[Y] = e^{at} y(t)$