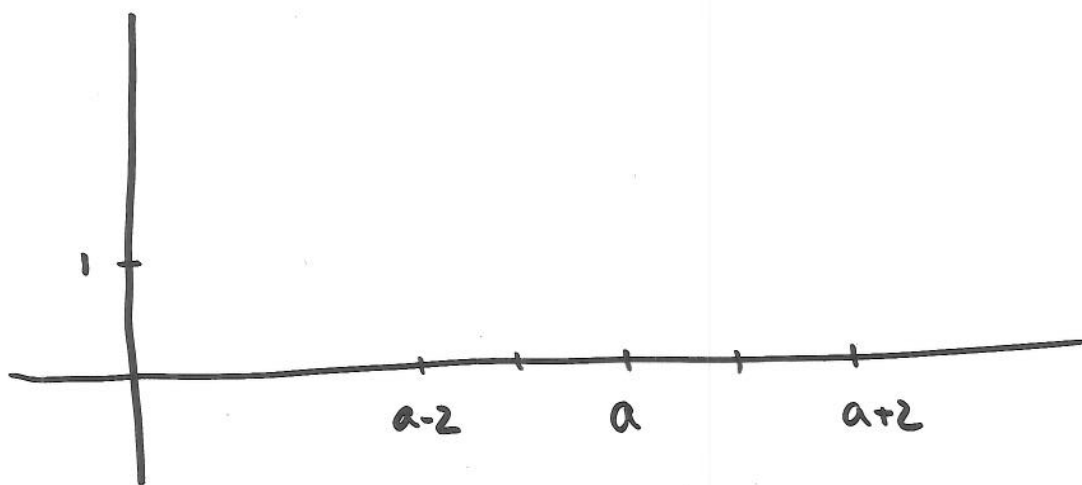


Sec 6.4 - Dirac Delta Function

We are going to define an impulse function.

That is a function this is non zero at only one point in its domain.

Define: $g_{a,\Delta t}(t) = \begin{cases} \frac{1}{2\Delta t}, & a-\Delta t < t \leq a+\Delta t \\ 0 & \text{otherwise} \end{cases}$



$$\int_{-\infty}^{\infty} g_{a,2}(t) dt =$$

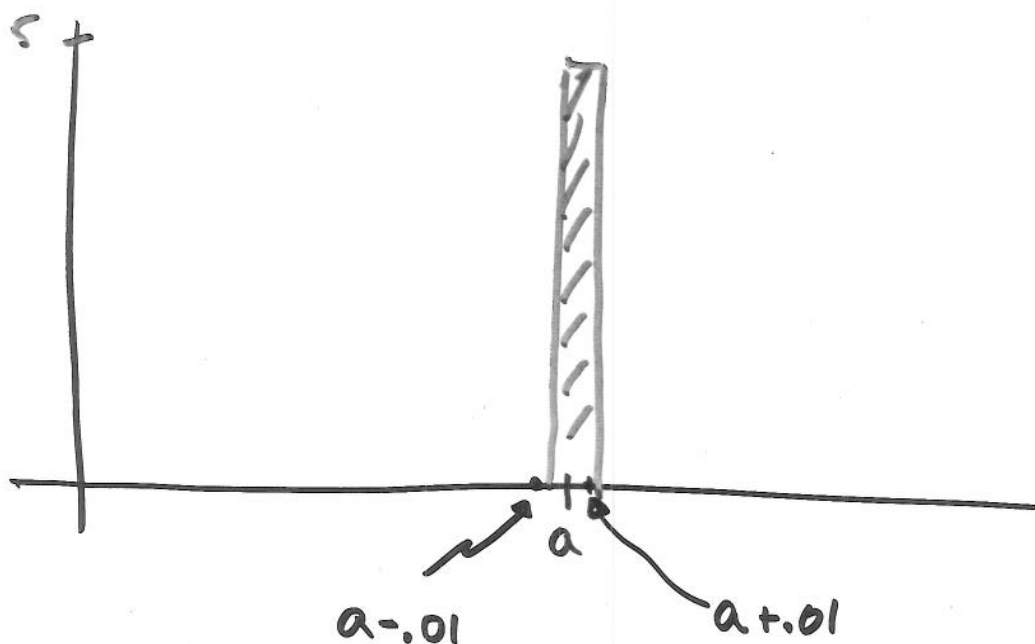
$$g_{a,2}(t) = \begin{cases} \frac{1}{4}, & a-2 \leq t \leq a+2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} g_{a,1}(t) dt =$$

$$g_{a,1}(t) = \begin{cases} \frac{1}{2}, & a-1 \leq t \leq a+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} g_{a,.1}(t) dt =$$

$$g_{a,.1}(t) = \begin{cases} 5, & a-.1 \leq t \leq a+.1 \\ 0 & \text{otherwise} \end{cases}$$



$$g_{a,.01}(t) = \begin{cases} 50, & a-.01 \leq t \leq a+.01 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} g_{a,.01}(t) dt = 1$$

$$\lim_{\Delta t \rightarrow 0} g_{a,\Delta t}(t) = \delta_a(t) \quad \text{Dirac Delta Function}$$

Recall the Mean Value Theorem for Integrals

$$\int_a^b f(x) dx = f(c)(b-a) \quad \text{for some } c \in [a, b]$$

Consider

$$\int_{-\infty}^{\infty} \delta_a(t) f(t) dt = \lim_{\Delta t \rightarrow 0} \int_{a-\Delta t}^{a+\Delta t} \frac{1}{2\Delta t} f(t) dt$$

$$= \lim_{\Delta t \rightarrow 0} \frac{1}{2\Delta t} \int_{a-\Delta t}^{a+\Delta t} f(t) dt$$

$$= \lim_{\Delta t \rightarrow 0} \frac{1}{2\Delta t} f(c) \cdot (a+\Delta t - (a-\Delta t))$$

$$c \in [a-\Delta t, a+\Delta t]$$

$$= \lim_{\Delta t \rightarrow 0} \frac{1}{2\Delta t} f(c) \cdot 2\Delta t$$

$$= \lim_{\Delta t \rightarrow 0} f(c) \quad c \in [a-\Delta t, a+\Delta t]$$

$$= f(a)$$

4

Properties of Dirac Delta Function $\delta_a(t)$

$$(i) \quad \delta_a(t) = 0 \quad \text{for all } t \neq a$$

$$(ii) \quad \int_{-\infty}^{\infty} \delta_a(t) dt = 1$$

$$(iii) \quad \int_{-\infty}^{\infty} \delta_a(t) f(t) dt = f(a)$$

Laplace Transform of $\delta_a(t)$

$$\mathcal{L}[\delta_a(t)] = \int_0^{\infty} \delta_a(t) e^{-st} dt = e^{-as}$$

$$\mathcal{L}[\delta_a(t)] = e^{-as}$$

Sec 6.4 #3

(5)

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 5y = \delta_3(t), \quad y(0)=1, \quad y'(0)=1$$

$$s^2 \mathcal{L}[y] - s y(0) - y'(0) + 2[s \mathcal{L}[y] - y(0)] + 5 \mathcal{L}[y] = e^{-3s}$$

$$s^2 \mathcal{L}[y] - s - 1 + 2s \mathcal{L}[y] - 2 \cdot 1 + 5 \mathcal{L}[y] = e^{-3s}$$

$$(s^2 + 2s + 5) \mathcal{L}[y] - s - 1 - 2 = e^{-3s}$$

$$(s^2 + 2s + 5) \mathcal{L}[y] = s + 3 + e^{-3s}$$

$$\mathcal{L}[y] = \frac{s+3}{s^2+2s+5} + \frac{e^{-3s}}{s^2+2s+5}$$

$$\mathcal{L}[y] = \frac{s+3}{(s+1)^2+4} + \frac{e^{-3s}}{(s+1)^2+4}$$

$$= \frac{s+1+2}{(s+1)^2+2^2} + \frac{1}{2} \frac{2e^{-3s}}{(s+1)^2+2^2}$$

$$= \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2} + \frac{1}{2} \frac{2e^{-3s}}{(s+1)^2+2^2}$$

$$y(t) = e^{-t} \cos 2t + e^{-t} \sin 2t + \frac{1}{2} u_3(t) e^{-(t-3)} \sin(2(t-3))$$

Sec 6.3 #33

$$\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 9y = 20 u_2(t) \sin(t-2) \quad y(0)=1, y'(0)=2$$

$$\mathcal{L}\left[\frac{d^2 y}{dt^2}\right] + 4 \mathcal{L}\left[\frac{dy}{dt}\right] + 9 \mathcal{L}[y] = 20 \mathcal{L}[u_2(t) \sin(t-2)]$$

$$s^2 \mathcal{L}[y] - s y(0) - y'(0) + 4[s \mathcal{L}[y] - y(0)] + 9 \mathcal{L}[y] = \frac{20 e^{-2s}}{s^2 + 1}$$

$$s^2 \mathcal{L}[y] - s - 2 + 4s \mathcal{L}[y] - 4 + 9 \mathcal{L}[y] = \frac{20 e^{-2s}}{s^2 + 1}$$

$$(s^2 + 4s + 9) \mathcal{L}[y] - s - 6 = \frac{20 e^{-2s}}{s^2 + 1}$$

$$s^2 + 4s + 9 \mathcal{L}[y] = s + 6 + \frac{20 e^{-2s}}{s^2 + 1}$$

$$\mathcal{L}[y] = \frac{s+6}{s^2+4s+9} + \frac{20 e^{-2s}}{(s^2+4s+9)(s^2+1)}$$

$$\mathcal{L}[y] = \frac{s+2+4}{(s+2)^2+5} + \frac{(-s+2)e^{-2s}}{s^2+1} + \frac{(s+2)e^{-2s}}{(s+2)^2+5}$$

$$\mathcal{L}[y] = \frac{s+2}{(s+2)^2+5} + \frac{1}{\sqrt{5}} \frac{4\sqrt{5}}{(s+2)^2+5} - \frac{se^{-2s}}{s^2+1} + \frac{2e^{-2s}}{s^2+1} + \frac{(s+2)e^{-2s}}{(s+2)^2+5}$$

$$y(t) = e^{-2t} \cos \sqrt{5} t + \frac{4}{\sqrt{5}} e^{-2t} \sin \sqrt{5} t$$

Partial Fractions

$$\frac{20}{(s^2+4s+9)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+9}$$

we find $A = -1, B = 2, C = 1, D = 2$

$$\frac{20}{(s^2+4s+9)(s^2+1)} = \frac{-s+2}{s^2+1} + \frac{s+2}{s^2+4s+9}$$

Table 6.1

Frequently Encountered Laplace Transforms.

$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$	$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$
$y(t) = e^{at}$	$Y(s) = \frac{1}{s-a} \quad (s > a)$	$y(t) = t^n$	$Y(s) = \frac{n!}{s^{n+1}} \quad (s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \frac{\omega}{s^2 + \omega^2}$	$y(t) = \cos \omega t$	$Y(s) = \frac{s}{s^2 + \omega^2}$
$y(t) = e^{at} \sin \omega t$	$Y(s) = \frac{\omega}{(s-a)^2 + \omega^2}$	$y(t) = e^{at} \cos \omega t$	$Y(s) = \frac{s-a}{(s-a)^2 + \omega^2}$
$y(t) = t \sin \omega t$	$Y(s) = \frac{2\omega s}{(s^2 + \omega^2)^2}$	$y(t) = t \cos \omega t$	$Y(s) = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$y(t) = u_a(t)$	$Y(s) = \frac{e^{-as}}{s} \quad (s > 0)$	$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$

Table 6.2

Rules for Laplace Transforms:

Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants α and a .

Rule for Laplace Transform	Rule for Inverse Laplace Transform
$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) = sY(s) - y(0)$	
$\mathcal{L}[y + w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$	$\mathcal{L}^{-1}[Y + W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$
$\mathcal{L}[\alpha y] = \alpha \mathcal{L}[y] = \alpha Y(s)$	$\mathcal{L}^{-1}[\alpha Y] = \alpha \mathcal{L}^{-1}[Y] = \alpha y(t)$
$\mathcal{L}[u_a(t)y(t-a)] = e^{-as} \mathcal{L}[y] = e^{-as} Y(s)$	$\mathcal{L}^{-1}[e^{-as} Y] = u_a(t)y(t-a)$
$\mathcal{L}[e^{at}y(t)] = Y(s-a)$	$\mathcal{L}^{-1}[Y(s-a)] = e^{at} \mathcal{L}^{-1}[Y] = e^{at} y(t)$