Sec 6.3 Homework

34.) $\frac{d^2y}{dt^2} + 3y = w(t) \qquad y(0) = 2 \qquad y'(0) = 0$

$$w(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

$$\mathcal{L}[w] = \int_0^\infty w(t) e^{-st} dt = \int_0^1 t\, e^{-st} dt + \int_1^\infty e^{-st} dt$$

$u = t \quad dv = e^{-st} dt$

$du = dt \quad v = -\frac{1}{s} e^{-st}$

$$= -\frac{t}{s} e^{-st} \Big|_0^1 - \int_0^1 -\frac{1}{s} e^{-st} dt + \lim_{b \to \infty} \int_1^b e^{-st}$$

$$= -\frac{e^{-s}}{s} - 0 + \frac{1}{s} \int_0^1 e^{-st} dt + \lim_{b \to \infty} -\frac{1}{s} e^{-st} \Big|_1^b$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s}\left(-\frac{1}{s} e^{-st} \Big|_0^1 \right) + \lim_{b \to \infty} -\frac{1}{s} e^{-sb} - -\frac{1}{s} e^{-s}$$

$$= -\frac{e^{-s}}{s} - \frac{1}{s^2} e^{-s} - \frac{-1}{s^2} + 0 + \frac{1}{s} e^{-s}$$

$$\boxed{\mathcal{L}[w] = \frac{1}{s^2}\left(1 - e^{-s}\right)}$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2} + 3y\right] = \mathcal{L}[w]$$

$$s^2 \mathcal{L}[y] - s[y(0)] - y'(0) + 3\mathcal{L}[y] = \frac{1}{s^2}\left(1 - e^{-s}\right)$$

$$(s^2 + 3)\mathcal{L}[y] - 2s = \frac{1}{s^2}\left(1 - e^{-s}\right)$$

Sec 6.3 Homework

(continued)

#34) $\mathcal{L}[y] = \frac{2s}{s^2+3} + \frac{1}{s^2(s^2+3)} - \frac{e^{-s}}{s^2(s^2+3)}$

$$\mathcal{L}[y] = \frac{2\cdot s}{s^2+3} + \frac{1}{3}\cdot\frac{1}{s^2} - \frac{1}{3}\cdot\frac{1}{s^2+3}$$

$$-\frac{1}{3}\frac{e^{-s}}{s^2} + \frac{1}{3}\frac{e^{-s}}{s^2+3}$$

$$\mathcal{L}[y] = 2\cdot\frac{s}{s^2+3} + \frac{1}{3}\cdot\frac{1}{s^2} - \frac{1}{3}\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}$$

$$-\frac{1}{3}\cdot\frac{e^{-s}}{s^2} + \frac{1}{3\sqrt{3}}\cdot\frac{e^{-s}}{s^2+(\sqrt{3})^2}$$

$$y(t) = 2\cos\sqrt{3}t + \frac{1}{3}t - \frac{1}{3\sqrt{3}}\cdot\sin(\sqrt{3}t)$$

$$-\frac{1}{3}u_1(t)(t-1) + \frac{1}{3\sqrt{3}}u_1(t)\sin\sqrt{3}(t-1)$$

$$= 2\cos\sqrt{3}t + \frac{1}{3}\left(t - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)\right)$$

$$-\frac{1}{3}u_1(t)\left[(t-1) - \frac{1}{\sqrt{3}}\sin\sqrt{3}(t-1)\right]$$

Sec 6.3 Homework

#34 continued

$$\frac{1}{s^2(s^2+3)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+3}$$

$$1 = (As+B)(s^2+3) + (Cs+D)s^2$$

$$1 = As^3 + 3As + Bs^2 + 3B + Cs^3 + Ds^2$$

$$1 = (A+C)s^3 + (B+D)s^2 + 3As + 3B$$

$3B = 1 \Rightarrow B = \frac{1}{3}$

$B+D = 0 \Rightarrow D = -B = -\frac{1}{3}$

$3A = 0 \Rightarrow A = 0$

$A+C = 0 \Rightarrow C = 0$

$$\frac{1}{s^2(s^2+3)} = \frac{\frac{1}{3}}{s^2} + \frac{-\frac{1}{3}}{s^2+3}$$