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Sec 6.2 Second-Order Equations

Find the Laplace Transform for $\sin(\omega t)$ and $\cos(\omega t)$

$$\mathcal{L}[\sin(\omega t)] = \int_0^\infty \underbrace{\sin(\omega t)}_{u}\,\underbrace{e^{-st}\,dt}_{dv}$$

$du = \omega\cos(\omega t)\,dt \qquad v = -\frac{1}{s}e^{-st}$

$$\int_0^\infty \sin(\omega t)e^{-st}dt = \lim_{b\to\infty}\left[\sin(\omega t)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^b - \int_0^b \left(-\frac{1}{s}\right)e^{-st}\cdot\omega\cdot\cos(\omega t)dt\right]$$

$$= \lim_{b\to\infty}\left[\sin(\omega b)\left(-\frac{1}{s}e^{-sb}\right) - \sin(0)\left(-\frac{1}{s}\right)e^{0}\right]$$

$$+ \lim_{b\to\infty}\frac{\omega}{s}\int_0^b \underbrace{\cos(\omega t)}_{u}\,\underbrace{e^{-st}dt}_{dv}$$

$du = -\omega\sin(\omega t)dt \qquad v = -\frac{1}{s}e^{-st}$

$$\int_0^\infty \sin(\omega t)e^{-st}dt = \lim_{b\to\infty}\left[\frac{\omega}{s}\cdot\cos(\omega t)\left(-\frac{1}{s}\right)e^{-st}\Big|_0^b\right]$$

$$- \lim_{b\to\infty}\frac{\omega}{s}\int_0^b\left(-\frac{1}{s}\right)e^{-st}(-\omega)\sin(\omega t)dt$$

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$$\int_0^\infty \sin(\omega t)e^{-st}dt = \lim_{b\to\infty}\left[\frac{\omega}{s}\cos(\omega b)\left(-\frac{1}{s}\right)e^{-sb} - \frac{\omega}{s}\cos(0)\left(-\frac{1}{s}\right)e^{0}\right]$$

$$- \lim_{b\to\infty}\frac{\omega^2}{s^2}\int_0^b \sin(\omega t)e^{-st}dt$$

$$\int_0^\infty \sin(\omega t)e^{-st}dt = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\int_0^\infty \sin(\omega t)e^{-st}dt$$

$$\int_0^\infty \sin(\omega t)e^{-st}dt + \frac{\omega^2}{s^2}\int_0^\infty \sin(\omega t)e^{-st}dt = \frac{\omega}{s^2}$$

$$\left(1+\frac{\omega^2}{s^2}\right)\int_0^\infty \sin(\omega t)e^{-st}dt = \frac{\omega}{s^2}$$

$$\frac{s^2+\omega^2}{s^2}\int_0^\infty \sin(\omega t)e^{-st}dt = \frac{\omega}{s^2}$$

$$\int_0^\infty \sin(\omega t)e^{-st}dt = \frac{\omega}{s^2}\cdot\frac{s^2}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}$$

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$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

Similarly, we can derive

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$$

One more fact

If $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$

Then
$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s\mathcal{L}\left[\frac{dy}{dt}\right] - y'(0)$$
$$= s\left[s\mathcal{L}[y] - y(0)\right] - y'(0)$$
$$= s^2\mathcal{L}[y] - s\,y(0) - y'(0)$$

We can now derive the formula

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

using another approach

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Recall $\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad y(0)=0 \quad y'(0)=\omega$

try $y(t) = e^{st}$

$$s^2e^{st} + \omega^2e^{st} = 0$$
$$s^2+\omega^2 = 0 \Rightarrow s = \pm\sqrt{-\omega^2} = i\omega$$

$y(t) = e^{i\omega t}$

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We can demonstrate that $\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$

through a similar method since

$y(t) = \cos(\omega t)$ is the solution of

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad y(0) = 1 \quad y'(0) = 0$$

(Try this at home)

Alternatively, since $\cos(\omega t) = \frac{1}{\omega}\frac{d}{dt}\sin(\omega t)$

$$\mathcal{L}[\cos(\omega t)] = \mathcal{L}\left[\frac{1}{\omega}\frac{d}{dt}\sin(\omega t)\right]$$

$$= \frac{1}{\omega}\mathcal{L}\left[\frac{d}{dt}\sin(\omega t)\right]$$

$$= \frac{1}{\omega}\cdot\left[s\,\mathcal{L}[\sin(\omega t)] - \sin(0)\right]$$

$$= \frac{s}{\omega}\cdot\mathcal{L}[\sin(\omega t)]$$

$$= \frac{s}{\omega}\cdot\frac{\omega}{s^2+\omega^2}$$

$$= \frac{s}{s^2+\omega^2}$$

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Example: Solve $\frac{d^2y}{dt^2} + 4y = 3\cos(t) \qquad y(0) = 0, \quad y'(0) = 0$

$$\mathcal{L}\left[\frac{d^2y}{dt^2} + 4y\right] = \mathcal{L}[3\cos(t)]$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + \mathcal{L}[4y] = 3\cdot\frac{s}{s^2+1}$$

$$s^2\mathcal{L}[y] - s\,y(0) - y'(0) + 4\mathcal{L}[y] = 3\cdot\frac{s}{s^2+1}$$

$$(s^2+4)\mathcal{L}[y] = \frac{3s}{s^2+1}$$

$$\mathcal{L}[y] = \frac{3s}{(s^2+1)(s^2+4)}$$

$$\boxed{\frac{3s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \Rightarrow \begin{array}{l} A=1,\ B=0 \\ C=-1,\ D=0 \end{array}}$$

$$y = \mathcal{L}^{-1}[\mathcal{L}[y]] = \mathcal{L}^{-1}\left[\frac{3s}{(s^2+1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]$$

$y(t) =$

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One more rule that is needed to use Laplace Transforms to solve 2nd order differential eqns.

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{at}f(t)\,e^{-st}\,dt$$

$$= \int_{t=0}^{t=\infty} f(t)\,e^{-(s-a)t}\,dt$$

Let $u = s-a$

$$= \int_{t=0}^{\infty} f(t)\,e^{-ut}\,dt$$

$$= F(u) = F(s-a)$$

$$\boxed{\mathcal{L}[e^{at}f(t)] = F(s-a)}$$

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Example: $\mathcal{L}[\cos(2t)] = \frac{s}{s^2+2^2} = \frac{s}{s^2+4}$

$$\mathcal{L}[e^{-3t}\cos(2t)] = \frac{s+3}{(s+3)^2+4} = \frac{s+3}{s^2+6s+9+4}$$

$$= \frac{s+3}{s^2+6s+13}$$

Example: Find $\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+5}\right]$

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Example $\mathcal{L}^{-1}\left[\dfrac{s+4}{s^2+4s+13}\right]$

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Example: Solve the IVP

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0 \qquad y(0)=1,\ y'(0)=0$$

try $y(t) = e^{st} \Rightarrow s^2+4s+13=0$

$$s = \frac{-4 \pm \sqrt{16-4\cdot 13}}{2}$$

$$s = -2 \pm 3i$$

$$y(t) = e^{(-2+3i)t}$$

$$y(t) = e^{-2t}\left(\cos(3t) + i\sin(3t)\right)$$

two real solutions $y_1(t) = e^{-2t}\cos(3t)$

$y_2(t) = e^{-2}\sin(3t)$

$$y(t) = k_1 e^{-2t}\cos(3t) + k_2 e^{-2t}\sin(3t)$$

$$y'(t) = k_1 e^{-2t}\left(-2\cos(3t) - 3\sin(3t)\right) + k_2 e^{-2t}\left(-2\sin(3t) + 3\cos(3t)\right)$$

$$y(0) = 1 = k_1$$

$$y'(0) = 0 = k_1(-2) + k_2(3) \Rightarrow k_1 = 1,\ k_2 = \frac{2}{3}$$

$$\boxed{y(t) = e^{-2t}\cos(3t) + \frac{2}{3}e^{-2t}\sin(3t)}$$

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Solve $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = 0 \qquad y(0)=1,\ y'(0)=0$

Use Laplace Transforms

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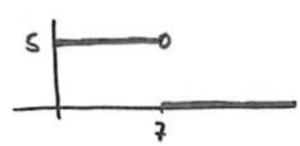
Oscillator with Discontinuous Forcing

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = h(t) \qquad y(0) = y'(0) = 0$$

$$h(t) = \begin{cases} 5, & t < 7 \\ 0, & t \geq 7 \end{cases}$$

$$h(t) = 5\left(1 - u_7(t)\right)$$



$$\mathcal{L}\left[\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y\right] = \mathcal{L}[h(t)]$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 2\mathcal{L}\left[\frac{dy}{dt}\right] + 5\mathcal{L}[y] = \mathcal{L}\left[5(1-u_7(t))\right]$$

$$s^2\mathcal{L}[y] - s\cdot y(0) - y'(0) + 2\left(s\mathcal{L}[y] - y(0)\right) + 5\mathcal{L}[y]$$

$= 5(\mathcal{L}[1] - \mathcal{L}[u_7$

$$(s^2+2s+5)\mathcal{L}[y] = 5\left(\frac{1}{s} - \frac{e^{-7s}}{s}\right)$$

$$\mathcal{L}[y] = \frac{5}{s(s^2+2s+5)} - \frac{5e^{-7s}}{s(s^2+2s+5)}$$

Partial Fractions

$$\frac{5}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5} \Rightarrow \begin{matrix} A=1 \\ B=-1 \\ C=-2 \end{matrix}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{5}{s(s^2+2s+5)} - \frac{5e^{-7s}}{s(s^2+2s+5)}\right]$$

Side problem

$$\frac{5}{s(s^2+2s+5)} = \frac{1}{s} - \frac{s+2}{s^2+2s+5} = \frac{1}{s} - \frac{s+2}{s^2+2s+1+4}$$

$$= \frac{1}{s} - \frac{s+2}{(s+1)^2+2^2} = \frac{1}{s} - \frac{s+1+1}{(s+1)^2+2^2}$$

$$= \frac{1}{s} - \frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}$$

$\mathcal{L}[\sin(2t)] = \frac{2}{s^2+2^2}$	$\mathcal{L}[\cos(2t)] = \frac{s}{s^2+2^2}$
$\mathcal{L}[e^{-t}\sin(2t)] = \frac{2}{(s+1)^2+2^2}$	$\mathcal{L}[e^{-t}\cos(2t)] = \frac{s+1}{(s+1)^2+2^2}$

$$\mathcal{L}^{-1}\left[\frac{5}{s(s^2+2s+5)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right]$$

$$= 1 - e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)$$

$$\mathcal{L}^{-1}\left[\frac{5e^{-7s}}{s(s^2+2s+5)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{e^{-7s}}{s}\right] - \mathcal{L}^{-1}\left[\frac{e^{-7s}(s+1)}{(s+1)^2+2^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-7s}\cdot 1}{(s+1)^2+2^2}\right]$$

$$= u_7(t) - u_7(t)\,e^{-(t-7)}\cos(2(t-7))$$

$$- \frac{1}{2}u_7(t)\,e^{-(t-7)}\sin(2(t-7))$$

$$= u_7(t)\left[1 - e^{-(t-7)}\cos(2(t-7)) - \frac{1}{2}e^{-(t-7)}\sin(2(t-7))\right]$$

Back to original problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = h(t) \qquad y(0) = y'(0) = 0$$

$$y(t) = 1 - e^{-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right)$$

$$- u_7(t)\left[1 - e^{-(t-7)}\left[\cos(2(t-7)) + \frac{1}{2}\sin(2(t-7))\right]\right]$$