

Sec 6.2 Second-Order Equations

Find the Laplace Transform for $\sin(\omega t)$
and $\cos(\omega t)$

$$\mathcal{L}[\sin(\omega t)] = \int_0^{\infty} \underbrace{\sin(\omega t)}_u \underbrace{e^{-st}}_{dv} dt$$

$$du = \omega \cos(\omega t) dt \quad v = -\frac{1}{s} e^{-st}$$

$$\int_0^{\infty} \sin(\omega t) e^{-st} dt = \lim_{b \rightarrow \infty} \left[\sin(\omega t) \left(-\frac{1}{s} e^{-st} \right) \right]_0^b - \int_0^b \left(-\frac{1}{s} \right) e^{-st} \cdot \omega \cos(\omega t) dt$$

$$= \lim_{b \rightarrow \infty} \left[\sin(\omega b) \left(-\frac{1}{s} e^{-sb} \right) - \sin(0) \left(-\frac{1}{s} \right) e^0 \right]$$

$$+ \lim_{b \rightarrow \infty} \frac{\omega}{s} \int_0^b \underbrace{\cos(\omega t)}_u \underbrace{e^{-st}}_{dv} dt$$

$$du = -\omega \sin(\omega t) dt \quad v = -\frac{1}{s} e^{-st}$$

$$\int_0^{\infty} \sin(\omega t) e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{\omega}{s} \cdot \cos(\omega t) \left(-\frac{1}{s} \right) e^{-st} \right]_0^b$$

$$- \lim_{b \rightarrow \infty} \frac{\omega}{s} \int_0^b \left(-\frac{1}{s} \right) e^{-st} (-\omega) \sin(\omega t) dt$$

$$\int_0^{\infty} \sin(\omega t) e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{\omega}{s} \cos(\omega b) \left(-\frac{1}{s} \right) e^{-sb} - \frac{\omega}{s} \cos(0) \left(-\frac{1}{s} \right) e^0 \right]$$

$$- \lim_{b \rightarrow \infty} \frac{\omega^2}{s^2} \int_0^b \sin(\omega t) e^{-st} dt$$

$$\int_0^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{s^2} - \frac{\omega^2}{s^2} \int_0^{\infty} \sin(\omega t) e^{-st} dt$$

$$\int_0^{\infty} \sin(\omega t) e^{-st} dt + \frac{\omega^2}{s^2} \int_0^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{s^2}$$

$$\left(1 + \frac{\omega^2}{s^2} \right) \int_0^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{s^2}$$

$$\frac{s^2 + \omega^2}{s^2} \int_0^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{s^2}$$

$$\int_0^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{s^2} \cdot \frac{s^2}{s^2 + \omega^2} = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$$

similarly, we can derive

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}$$

One more fact

$$\text{If } \mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

$$\begin{aligned} \text{Then } \mathcal{L}\left[\frac{d^2y}{dt^2}\right] &= s\mathcal{L}\left[\frac{dy}{dt}\right] - y'(0) \\ &= s[s\mathcal{L}[y] - y(0)] - y'(0) \\ &= s^2\mathcal{L}[y] - sy(0) - y'(0) \end{aligned}$$

We can now derive the formula

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$$

using another approach

$$\text{Recall } \frac{d^2y}{dt^2} + \omega^2 y = 0 \quad y(0)=0 \quad y'(0)=\omega$$

$$\text{try } y(t) = e^{st}$$

$$s^2 e^{st} + \omega^2 e^{st} = 0$$

$$s^2 + \omega^2 = 0 \Rightarrow s = \pm \sqrt{-\omega^2} = \pm i\omega$$

$$y(t) = e^{i\omega t}$$

We can demonstrate that $\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}$

through a similar method since

$y(t) = \cos(\omega t)$ is the solution of

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0 \quad y(0) = 1 \quad y'(0) = 0$$

(Try this at home)

Alternatively, since $\cos(\omega t) = \frac{1}{\omega} \frac{d}{dt} \sin(\omega t)$

$$\begin{aligned} \mathcal{L}[\cos(\omega t)] &= \mathcal{L}\left[\frac{1}{\omega} \frac{d}{dt} \sin(\omega t)\right] \\ &= \frac{1}{\omega} \mathcal{L}\left[\frac{d}{dt} \sin(\omega t)\right] \\ &= \frac{1}{\omega} \left[s \mathcal{L}[\sin(\omega t)] - \sin(0) \right] \\ &= \frac{s}{\omega} \cdot \mathcal{L}[\sin(\omega t)] \\ &= \frac{s}{\omega} \cdot \frac{\omega}{s^2 + \omega^2} \\ &= \frac{s}{s^2 + \omega^2} \end{aligned}$$

Example: Solve $\frac{d^2 y}{dt^2} + 4y = 3\cos(t)$ $y(0) = 0$
 $y'(0) = 0$

$$\mathcal{L}\left[\frac{d^2 y}{dt^2} + 4y\right] = \mathcal{L}[3\cos(t)]$$

$$\mathcal{L}\left[\frac{d^2 y}{dt^2}\right] + \mathcal{L}[4y] = 3 \cdot \frac{s}{s^2 + 1}$$

$$s^2 \mathcal{L}[y] - sy(0) - y'(0) + 4 \mathcal{L}[y] = 3 \cdot \frac{s}{s^2 + 1}$$

$$(s^2 + 4) \mathcal{L}[y] = \frac{3s}{s^2 + 1}$$

$$\mathcal{L}[y] = \frac{3s}{(s^2 + 1)(s^2 + 4)}$$

$\frac{3s}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4} \Rightarrow C = -1, D = 0$

$$y = \mathcal{L}^{-1}[\mathcal{L}[y]] = \mathcal{L}^{-1}\left[\frac{3s}{(s^2 + 1)(s^2 + 4)}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2 + 4}\right]$$

$$y(t) =$$

One more rule that is needed to use Laplace Transforms to solve 2nd order differential eqns.

$$\mathcal{L}[e^{at} f(t)] = \int_0^{\infty} e^{at} f(t) e^{-st} dt$$

$$= \int_{t=0}^{t=\infty} f(t) e^{-(s-a)t} dt$$

Let $u = s - a$

$$= \int_{t=0}^{\infty} f(t) e^{-ut} dt$$

$$= F(u) = F(s-a)$$

$$\boxed{\mathcal{L}[e^{at} f(t)] = F(s-a)}$$

Example: $\mathcal{L}[\cos(2t)] = \frac{s}{s^2+2^2} = \frac{s}{s^2+4}$

$$\mathcal{L}[e^{-3t} \cos(2t)] = \frac{s+3}{(s+3)^2+4} = \frac{s+3}{s^2+6s+9+4} = \frac{s+3}{s^2+6s+13}$$

Example: Find $\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+5}\right]$

Example

$$\mathcal{L}^{-1} \left[\frac{s+4}{s^2+4s+13} \right]$$

Example: Solve the IVP

$$\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 13y = 0 \quad y(0)=1, y'(0)=0$$

$$\text{try } y(t) = e^{st} \Rightarrow s^2 + 4s + 13 = 0$$

$$s = \frac{-4 \pm \sqrt{16 - 4 \cdot 13}}{2}$$

$$s = -2 \pm 3i$$

$$y_1(t) = e^{(-2+3i)t}$$

$$y_2(t) = e^{-2t} (\cos(3t) + i \sin(3t))$$

two real solutions $y_1(t) = e^{-2t} \cos(3t)$

$$y_2(t) = e^{-2t} \sin(3t)$$

$$y(t) = K_1 e^{-2t} \cos(3t) + K_2 e^{-2t} \sin(3t)$$

$$y'(t) = K_1 e^{-2t} (-2 \cos(3t) - 3 \sin(3t)) + K_2 e^{-2t} (-2 \sin(3t) + 3 \cos(3t))$$

$$y(0) = 1 = K_1 \Rightarrow K_1 = 1$$

$$y'(0) = 0 = K_1(-2) + K_2(3) \Rightarrow K_2 = \frac{2}{3}$$

$$y(t) = e^{-2t} \cos(3t) + \frac{2}{3} e^{-2t} \sin(3t)$$

Solve

$$\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 13y = 0$$

$$y(0) = 1$$

$$y'(0) = 0$$

Use Laplace Transforms

Oscillator with Discontinuous Forcing

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 5y = h(t) \quad y(0) = y'(0) = 0$$

$$h(t) = \begin{cases} 5, & t < \pi \\ 0, & t \geq \pi \end{cases}$$

$$h(t) = 5(1 - u_\pi(t))$$



$$\mathcal{L} \left[\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 5y \right] = \mathcal{L} [h(t)]$$

$$\mathcal{L} \left[\frac{d^2 y}{dt^2} \right] + 2 \mathcal{L} \left[\frac{dy}{dt} \right] + 5 \mathcal{L} [y] = \mathcal{L} [5(1 - u_\pi(t))]$$

$$s^2 \mathcal{L} [y] - s y(0) - y'(0) + 2(s \mathcal{L} [y] - y(0)) + 5 \mathcal{L} [y] = 5 \mathcal{L} [1] - \mathcal{L} [u_\pi]$$

$$(s^2 + 2s + 5) \mathcal{L} [y] = 5 \left(\frac{1}{s} - \frac{e^{-\pi s}}{s} \right)$$

$$\mathcal{L} [y] = \frac{5}{s(s^2 + 2s + 5)} - \frac{5e^{-\pi s}}{s(s^2 + 2s + 5)}$$

Partial Fractions

$$\frac{5}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5} \Rightarrow \begin{matrix} A=1 \\ B=-1 \\ C=-2 \end{matrix}$$

$$y(t) = \mathcal{L}^{-1} \left[\frac{5}{s(s^2+2s+5)} - \frac{5e^{-7s}}{s(s^2+2s+5)} \right]$$

Side problem

$$\frac{5}{s(s^2+2s+5)} = \frac{1}{s} - \frac{s+2}{s^2+2s+5} = \frac{1}{s} - \frac{s+2}{s^2+2s+1+4}$$

$$= \frac{1}{s} - \frac{s+2}{(s+1)^2+2^2} = \frac{1}{s} - \frac{s+1+1}{(s+1)^2+2^2}$$

$$= \frac{1}{s} - \frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}$$

$$\mathcal{L}[\sin(2t)] = \frac{2}{s^2+2^2}$$

$$\mathcal{L}[\cos(2t)] = \frac{s}{s^2+2^2}$$

$$\mathcal{L}[e^{-t}\sin(2t)] = \frac{2}{(s+1)^2+2^2}$$

$$\mathcal{L}[e^{-t}\cos(2t)] = \frac{s+1}{(s+1)^2+2^2}$$

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{5}{s(s^2+2s+5)} \right] &= \mathcal{L}^{-1} \left[\frac{1}{s} \right] - \mathcal{L}^{-1} \left[\frac{s+1}{(s+1)^2+2^2} \right] - \frac{1}{2} \mathcal{L}^{-1} \left[\frac{2}{(s+1)^2+2^2} \right] \\ &= 1 - e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t) \end{aligned}$$

$$\mathcal{L}^{-1} \left[\frac{5e^{-7s}}{s(s^2+2s+5)} \right]$$

$$= \mathcal{L}^{-1} \left[\frac{e^{-7s}}{s} \right] - \mathcal{L}^{-1} \left[\frac{e^{-7s}(s+1)}{(s+1)^2+2^2} \right] - \mathcal{L}^{-1} \left[\frac{e^{-7s}}{(s+1)^2+2^2} \right]$$

$$= u_7(t) - u_7(t)e^{-(t-7)}\cos(2(t-7))$$

$$- \frac{1}{2}u_7(t)e^{-(t-7)}\sin(2(t-7))$$

$$= u_7(t) \left[1 - e^{-(t-7)}\cos(2(t-7)) - \frac{1}{2}e^{-(t-7)}\sin(2(t-7)) \right]$$

Back to original problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = h(t) \quad y(0) = y'(0) = 0$$

$$\begin{aligned} y(t) &= 1 - e^{-t} \left(\cos(2t) + \frac{1}{2}\sin(2t) \right) \\ &\quad - u_7(t) \left[1 - e^{-(t-7)} \left(\cos(2(t-7)) + \frac{1}{2}\sin(2(t-7)) \right) \right] \end{aligned}$$