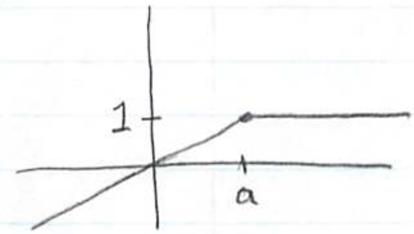
Sec 6.2 Homework

#3 Suppose $a \geq 0$. Compute the Laplace Transform of $g_a(t) = \begin{cases} t/a & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$



$$\mathcal{L}[g_a(t)] = \int_0^\infty g_a(t)\, e^{-st}\, dt$$

$$= \int_0^a g_a(t)\, e^{-st}\, dt + \int_a^\infty g_a(t)\, e^{-st}\, dt$$

$$= \frac{1}{a}\int_0^a t\, e^{-st}\, dt + \int_a^\infty e^{-st}\, dt$$

$u = t \quad du = dt$

$dv = e^{-st}dt$

$v = -\frac{1}{s}e^{-st}$

$$= \frac{1}{a}\left[t \cdot \frac{-1}{s} e^{-st} \Big|_0^a - \int_0^a -\frac{1}{s} e^{-st}\, dt \right] + \lim_{b \to \infty} \int_a^b e^{-st}\, dt$$

$$= \frac{1}{a}\left[\left[\frac{-a}{s} e^{-sa} - 0 \right] + \frac{-1}{s^2} e^{-st} \Big|_0^a \right] + \lim_{b \to \infty} -\frac{1}{s} e^{-st} \Big|_a^b$$

$$= \frac{-1}{s} e^{-sa} - \frac{1}{as^2} e^{-sa} + \frac{1}{as^2} + \overbrace{\lim_{b \to \infty} -\frac{1}{s} e^{-sb}}^{\to 0} + \frac{1}{s} e^{-sa}$$

$$\boxed{\mathcal{L}[g_a] = \frac{1}{as^2}\left(1 - e^{-as}\right)}$$