Sec 6.2 Laplace Transforms for Discontinuous Functions

$$\mathcal{L}(t) = \int_0^\infty t\, e^{-st}\, dt$$

$$= \lim_{b\to\infty} \int_0^b t\, e^{-st}\, dt$$

$u = t \qquad du = dt$

$dv = e^{-st} dt \qquad v = -\frac{1}{s} e^{-st}$

$$= \lim_{b\to\infty} t \cdot \frac{-1}{s} e^{-st} \Big|_0^b - \int_0^b -\frac{1}{s} e^{-st}\, dt$$

$$= \lim_{b\to\infty} \left[-\frac{b}{s} e^{-sb} - 0 \right] + \frac{1}{s} \int_0^b e^{-st}\, dt$$

$$= \frac{-1}{s} \lim_{b\to\infty} \frac{b}{e^{sb}} + \lim_{b\to\infty} \frac{-1}{s^2} e^{-st} \Big|_0^b$$

$$= -\frac{1}{s} \lim_{b\to\infty} \frac{1}{s e^{sb}} - \frac{1}{s^2} \lim_{b\to\infty} \left(e^{-sb} - e^{0} \right)$$

$=$

$$\mathcal{L}(t^2) = \int_0^\infty t^2 e^{-st}\, dt = \lim_{b\to\infty} \int_0^b t^2 e^{-st}\, dt$$

$$= \lim_{b\to\infty} t^2 \cdot \frac{-1}{s} e^{-st} \Big|_0^b - \int_0^b 2t \left(\frac{-1}{s} e^{-st} \right) dt$$

$$= \lim_{b\to\infty} \left[-\frac{b^2}{s} e^{-sb} + 0 \right] + \lim_{b\to\infty} \frac{2}{s} \int_0^b t e^{-st}\, dt$$

$$= -\frac{1}{s} \lim_{b\to\infty} \frac{b^2}{e^{sb}} + \frac{2}{s} \lim_{b\to\infty} \int_0^b t e^{-st}$$

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The Heavyside Function

For $a \geq 0$, let $u_a(t)$ denote

$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

$$\mathcal{L}[u_a(t)] = \int_0^\infty u_a(t)\, e^{-st}\, dt$$

$$= \int_0^a u_a(t) e^{-st} dt + \int_a^\infty u_a(t) e^{-st} dt$$

$$= 0 + \lim_{b\to\infty} \int_a^b e^{-st} dt$$

$$= \lim_{b\to\infty} -\frac{1}{s} e^{-st} \Big|_a^b$$

$$= \lim_{b\to\infty} -\frac{1}{s} e^{-sb} + \frac{1}{s} e^{-sa}$$

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$$\mathcal{L}[u_a(t)] = \frac{e^{-sa}}{s}$$

__Example__ $\mathcal{L}[u_3(t)] = \dfrac{e^{-3s}}{s}$

__Example__ $\mathcal{L}[2t^2 + 3t + 4] =$

Laplace Transforms

$y(t) = e^{at}$ $\qquad$ $Y(s) = \frac{1}{s-a}$, $s > a$

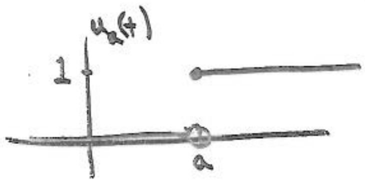
$y(t) = 1$ $\qquad$ $Y(s) = \frac{1}{s}$

$y(t) = t$ $\qquad$ $Y(s) = \frac{1}{s^2}$

$y(t) = t^2$ $\qquad$ $Y(s) = \frac{2}{t^3}$

$y(t) = t^n$ $\qquad$ $Y(s) = \frac{n!}{t^{n+1}}$

$y(t) = u_a(t)$ $\qquad$ $Y(s) = \frac{e^{-as}}{s}$ $\quad s > 0$



Properties of Laplace Transforms

$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$

$\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$

$\mathcal{L}[c \cdot f(t)] = c\,\mathcal{L}[f]$

$\mathcal{L}[u_a(t) f(t-a)] = e^{-as} \cdot \mathcal{L}[f]$

<u>Example</u> $\qquad \frac{dy}{dt} = -y + u_3(t) \qquad y(0) = 2$

<u>Old School - Using Methods from Chapter 1</u>

$$\frac{dy}{dt} = \begin{cases} -y & \text{if } t < 3 \\ -y+1 & \text{if } t \geq 3 \end{cases}$$

<u>step 1</u>: Solve IVP $\quad \frac{dy}{dt} = -y \qquad y(0) = -2$

$y(t) = ke^{-t} \qquad y(0) = 2 \Rightarrow k = 2$

$y(t) = 2e^{-t} \qquad 0 \leq t < 3$

<u>Note</u>: $y(3) = 2e^{-3}$

<u>step 2</u>: Solve IVP $\quad \frac{dy}{dt} = -y + 1 \qquad y(3) = 2e^{-3}$

$y_h(t) = Ke^{-t} \qquad y_p(t) = A \qquad$ solve for $A \Rightarrow A = 1$

$y(t) = Ke^{-t} + 1$

$y(3) = Ke^{-3} + 1 = 2e^{-3} \Rightarrow$

$Ke^{-3} = 2e^{-3} - 1$

$K = \underline{2e^{-3}-1} = 2 - e^3$

$\frac{dy}{dt} = -y + u_3(t) \qquad y(0) = 2$

$$\frac{dy}{dt} = \begin{cases} -y & \text{if } t < 3 \\ -y+1 & \text{if } t \geq 3 \end{cases} \qquad y(0) = 2$$

$$y(t) = \begin{cases} 2e^{-t} & t < 3 \\ (2-e^3)e^{-t} + 1 & t \geq 3 \end{cases}$$

$$y(t) = \begin{cases} 2e^{-t} & t < 3 \\ 2e^{-t} + 1 - e^{-(t-3)} & t \geq 3 \end{cases}$$

$$y(t) = 2e^{-t} + u_3(t)[1 - e^{-(t-3)}]$$

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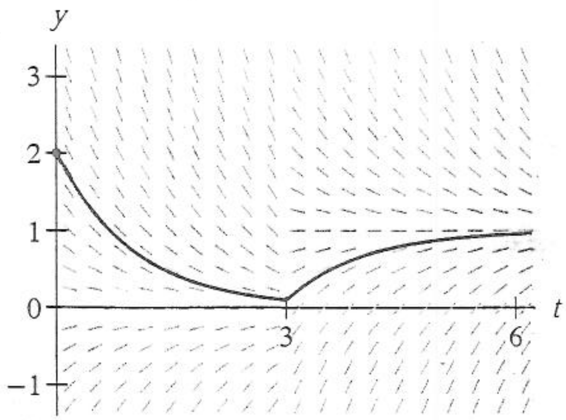

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Using Laplace Transforms to solve

$$\frac{dy}{dt} = -y + u_3(t) \qquad y(0) = 2$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \mathcal{L}\left[-y + u_3(t)\right]$$

$$s\mathcal{L}[y] - y(0) = \mathcal{L}[-y] + \mathcal{L}[u_3(t)]$$

$$s\mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{e^{-3s}}{s}$$

$$(s+1)\mathcal{L}[y] = 2 + \frac{e^{-3s}}{s}$$

$$\mathcal{L}[y] = \frac{2}{s+1} + \frac{e^{-3s}}{s(s+1)}$$

$$\mathcal{L}^{-1}[\mathcal{L}[y]] = \mathcal{L}^{-1}\left[\frac{2}{s+1} + \frac{e^{-3s}}{s(s+1)}\right]$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s(s+1)}\right]$$

$$y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[e^{-3s}\cdot\frac{1}{s(s+1)}\right]$$

$$\boxed{\frac{1}{s(s+1)} = \frac{a}{s} - \frac{b}{s+1} = \frac{1}{s} - \frac{1}{s+1}}$$

$$y(t) = 2e^{-t} + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s+1}\right]$$

$$y(t) = 2e^{-t} + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] - \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+1}\right]$$

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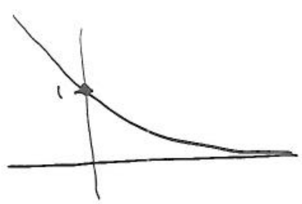
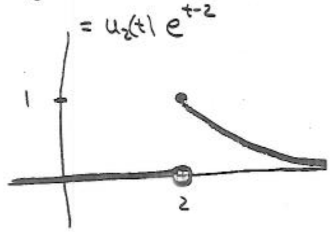
$$\boxed{\mathcal{L}[u_a(t) f(t-a)] = e^{-as}\mathcal{L}[f] = e^{-as}F(s)}$$

Example: If $f(t) = e^{-t}$, $g(t) = u_2(t) f(t-2)$
$= u_2(t) e^{t-2}$





Note: $g(2) = u_2(2) f(2-2) = 1\cdot f(0) = f(0) = 1$

$g(3) = g(2+1) = u_2(3) f(3-2) = 1\cdot f(1) = f(1)$

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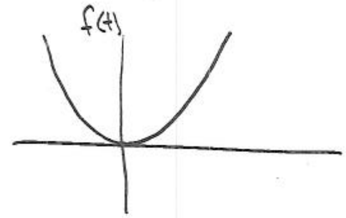
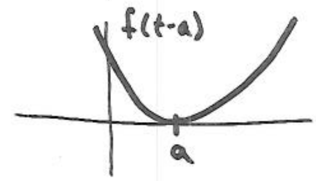
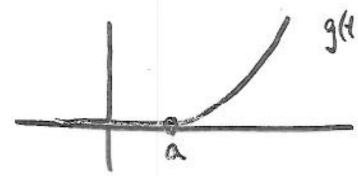
In general, if $g(t) = u_a(t) f(t-a)$

then $g(a) = u_a(a) f(a-a) = 1\cdot f(0) = f(0)$

$g(a+b) = u_a(a+b) f(a+b-a) = 1\cdot f(b) = f(b)$







$g(t) = u_a(t) f(t-a)$

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$$\mathcal{L}[u_a(t) f(t-a)] = \int_0^\infty u_a(t) f(t-a) e^{-st} dt$$

$$= \int_0^a u_a(t) f(t-a) e^{-st} dt + \int_a^\infty u_a(t) f(t-a) e^{-st} dt$$

$$= \int_a^\infty f(t-a) e^{-st} dt$$

change of variables

Let $u = t - a \Rightarrow t = u + a$

$du = dt$

Limits of integration : when $t = a$ $\quad u = a - a = 0$

$t = \infty \quad u = \infty - a = \infty$

$$= \int_0^\infty f(u) e^{-s(u+a)} du$$

$$= \int_0^\infty f(u) e^{-su} \cdot e^{-as} du$$

$$= e^{-as} \cdot \int_0^\infty f(u) e^{-su} du$$

$$\mathcal{L}[u_a(t) f[t-a]] = e^{-as} \mathcal{L}[f] = e^{-as} F(s)$$

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Examples $\quad f(t) = e^{2t} \qquad F(s) = \frac{1}{s-2}, \; s > 2$

$$\mathcal{L}[u_3(t) e^{2(t-3)}] = \frac{e^{-3s}}{s-2}$$

Example: $\mathcal{L}[u_1(t) e^{-3(t-1)}] =$

Example: $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+1}\right] =$

- back to page 10 to finish our problem