

Sec 6.2 Laplace Transforms for Discontinuous Functions

$$f(t) = \int_0^{\infty} t e^{-st} dt$$

$$= \lim_{b \rightarrow \infty} \int_0^b t e^{-st} dt$$

$$u = t \quad du = dt$$

$$dv = e^{-st} dt$$

$$v = -\frac{1}{s} e^{-st}$$

$$= \lim_{b \rightarrow \infty} t \cdot \frac{-1}{s} e^{-st} \Big|_0^b - \int_0^b -\frac{1}{s} e^{-st} \cdot dt$$

$$= \lim_{b \rightarrow \infty} \left[-\frac{b}{s} e^{-sb} - 0 \right] + \frac{1}{s} \int_0^b e^{-st} dt$$

$$= \frac{-1}{s} \lim_{b \rightarrow \infty} \frac{b}{e^{sb}} + \lim_{b \rightarrow \infty} \left. -\frac{1}{s^2} e^{-st} \right|_0^b$$

$$= -\frac{1}{s} \lim_{b \rightarrow \infty} \frac{1}{se^{sb}} - \frac{1}{s^2} \lim_{b \rightarrow \infty} (e^{-sb} - e^0)$$

=

$$f(t^2) = \int_0^{\infty} t^2 e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b t^2 e^{-st} dt$$

$$= \lim_{b \rightarrow \infty} t^2 \cdot \frac{-1}{s} e^{-st} \Big|_0^b - \int_0^b 2t \left(-\frac{1}{s} e^{-st} \right) dt$$

$$= \lim_{b \rightarrow \infty} \left[-\frac{b^2}{s} e^{-sb} + 0 \right] + \lim_{b \rightarrow \infty} \frac{2}{s} \int_0^b t e^{-st} dt$$

$$= -\frac{1}{s} \lim_{b \rightarrow \infty} \frac{b^2}{e^{sb}} + \frac{2}{s} \lim_{b \rightarrow \infty} \int_0^b t e^{-st} dt$$

The Heaviside Function

For $a \geq 0$, let $u_a(t)$ denote

$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



$$\begin{aligned} \mathcal{L}[u_a(t)] &= \int_0^{\infty} u_a(t) e^{-st} dt \\ &= \int_0^a u_a(t) e^{-st} dt + \int_a^{\infty} u_a(t) e^{-st} dt \\ &= 0 + \lim_{b \rightarrow \infty} \int_a^b e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \left. -\frac{1}{s} e^{-st} \right|_a^b \\ &= \lim_{b \rightarrow \infty} -\frac{1}{s} e^{-sb} + \frac{1}{s} e^{-sa} \end{aligned}$$

$$\mathcal{L}[u_a(t)] = \frac{e^{-sa}}{s}$$

Example $\mathcal{L}[u_3(t)] = \frac{e^{-3s}}{s}$

Example $\mathcal{L}[2t^2 + 3t + 4] =$

Laplace Transforms

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$$y(t) = e^{at} \quad Y(s) = \frac{1}{s-a}, \quad s > a$$

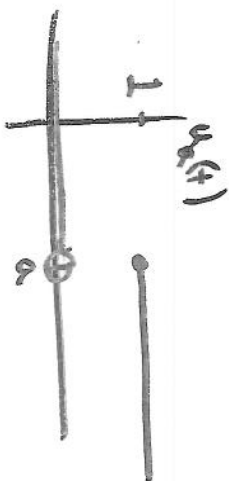
$$y(t) = 1 \quad Y(s) = \frac{1}{s}$$

$$y(t) = t \quad Y(s) = \frac{1}{s^2}$$

$$y(t) = t^2 \quad Y(s) = \frac{2}{s^3}$$

$$y(t) = t^n \quad Y(s) = \frac{n!}{s^{n+1}}$$

$$y(t) = u_a(t) \quad Y(s) = \frac{e^{-as}}{s} \quad s > 0$$



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Properties of Laplace Transforms

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$$

$$\mathcal{L}[c \cdot f(t)] = c \mathcal{L}[f]$$

$$\mathcal{L}[u_a(t)f(t-a)] = e^{-as} \cdot \mathcal{L}[f]$$

Example

$$\frac{dy}{dt} = -y + u_3(t) \quad y(0) = 2$$

Old School - Using Methods from Chapter 1

$$\frac{dy}{dt} = \begin{cases} -y & \text{if } t < 3 \\ -y+1 & \text{if } t \geq 3 \end{cases}$$

Step 1: Solve IVP $\frac{dy}{dt} = -y \quad y(0) = -2$

$$y(t) = Ke^{-t} \quad y(0) = 2 \Rightarrow K = 2$$

$$y(t) = 2e^{-t} \quad 0 \leq t < 3$$

Note: $y(3) = 2e^{-3}$

Step 2: Solve IVP $\frac{dy}{dt} = -y+1 \quad y(3) = 2e^{-3}$

$$y_h(t) = Ke^{-t} \quad y_p(t) = A \quad \text{solve for } A \Rightarrow A = 1$$

$$y(t) = Ke^{-t} + 1$$

$$y(3) = Ke^{-3} + 1 = 2e^{-3} \Rightarrow$$

$$Ke^{-3} = 2e^{-3} - 1$$

$$K = \frac{2e^{-3} - 1}{e^{-3}} = 2 - e^3$$

$$\frac{dy}{dt} = -y + u_3(t) \quad y(0) = 2$$

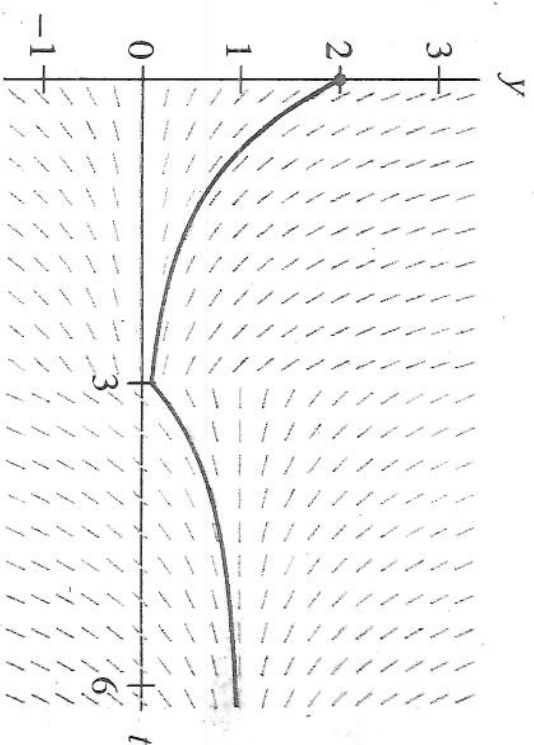
$$\frac{dy}{dt} = \begin{cases} -y & \text{if } t < 3 \\ -y+1 & \text{if } t \geq 3 \end{cases} \quad y(0) = 2$$

$$y(t) = \begin{cases} 2e^{-t} & t < 3 \\ (2 - e^3)e^{-t} + 1 & t \geq 3 \end{cases}$$

$$y(t) = \begin{cases} 2e^{-t} & t < 3 \\ 2e^{-t} + 1 - e^{-(t-3)} & t \geq 3 \end{cases}$$

$$y(t) = 2e^{-t} + u_3(t)[1 - e^{-(t-3)}]$$

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Using Laplace Transforms to solve

$$\frac{dy}{dt} = -y + u_3(t) \quad y(0) = 2$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \mathcal{L}[-y + u_3(t)]$$

$$s\mathcal{L}[y] - y(0) = \mathcal{L}[-y] + \mathcal{L}[u_3(t)]$$

$$s\mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{e^{-3s}}{s}$$

$$(s+1)\mathcal{L}[y] = 2 + \frac{e^{-3s}}{s}$$

$$\mathcal{L}[y] = \frac{2}{s+1} + \frac{e^{-3s}}{s(s+1)}$$

$$\mathcal{L}^{-1}[\mathcal{L}[y]] = \mathcal{L}^{-1}\left[\frac{2}{s+1} + \frac{e^{-3s}}{s(s+1)}\right]$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s(s+1)}\right]$$

$$y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s(s+1)}\right]$$

$$\boxed{\frac{1}{s(s+1)} = \frac{a}{s} - \frac{b}{s+1} = \frac{1}{s} - \frac{1}{s+1}}$$

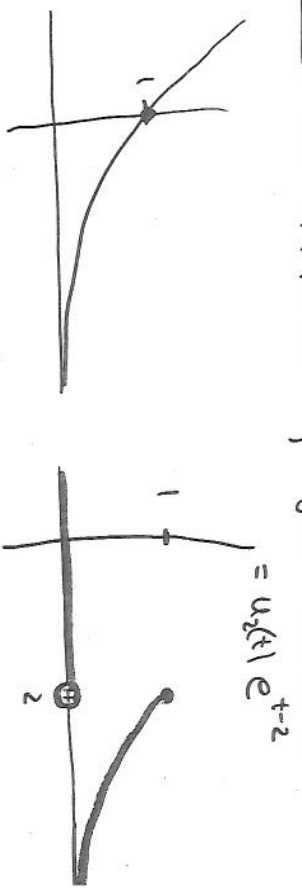
$$y(t) = 2e^{-t} + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s+1}\right]$$

$$y(t) = 2e^{-t} + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] - \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+1}\right]$$

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$$\mathcal{L}[u_a(t)f(t-a)] = e^{-as} \mathcal{L}[f] = e^{-as} F(s)$$

Example: If $f(t) = e^{-t}$, $g(t) = u_2(t)f(t-2)$



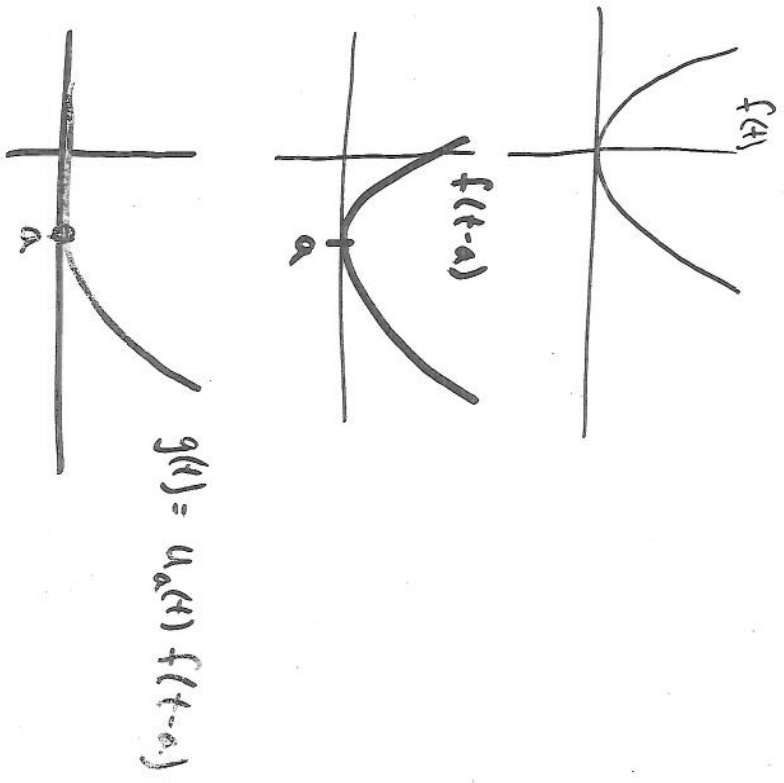
Note: $g(2) = u_2(2)f(2-2) = 1 \cdot f(0) = f(0) = 1$

$g(3) = g(2+1) = u_2(3)f(3-2) = 1 \cdot f(1) = f(1)$

In general, if $g(t) = u_a(t)f(t-a)$

Then $g(a) = u_a(a)f(a-a) = 1 \cdot f(0) = f(0)$

$g(a+b) = u_a(a+b)f(a+b-a) = 1 \cdot f(b) = f(b)$



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$$\mathcal{L}[u_a(t)f(t-a)] = \int_0^{\infty} u_a(t)f(t-a)e^{-st}dt$$

$$= \int_0^a u_a(t)f(t-a)e^{-st}dt + \int_a^{\infty} u_a(t)f(t-a)e^{-st}dt$$

$$= \int_a^{\infty} f(t-a)e^{-st}dt$$

change of variables

let $u = t - a \Rightarrow t = u + a$
 $du = dt$

Limits of integration: when $t = a$ $u = a - a = 0$
 $t = \infty$ $u = \infty - a = \infty$

$$= \int_0^{\infty} f(u)e^{-s(u+a)}du$$

$$= \int_0^{\infty} f(u)e^{-su} \cdot e^{-as}du$$

$$= e^{-as} \cdot \int_0^{\infty} f(u)e^{-su}du$$

$$\mathcal{L}[u_a(t)f(t-a)] = e^{-as} \mathcal{L}[f] = e^{-as} F(s)$$

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Example $f(t) = e^{2t}$ $F(s) = \frac{1}{s-2}, s > 2$

$$\mathcal{L}[u_3(t)e^{2(t-3)}] = \frac{e^{-3s}}{s-2}$$

Example: $\mathcal{L}[u_1(t)e^{-3(t-1)}] =$

Example: $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+1}\right] =$

- back to page 10 to finish our problem