Definition: The Laplace Transform function $Y(s)$ of the function $y(t)$ is defined

$$\mathcal{L}[y(t)] = Y(s) = \int_0^\infty e^{-st} \cdot y(t)\, dt$$

Example: $y(t) = 1$

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{a\to\infty} \int_0^a e^{-st} dt$$

$$= \lim_{a\to\infty} -\frac{1}{s} e^{-st} \Big|_0^a = \lim_{a\to\infty} -\frac{1}{s}\left[e^{-a\cdot s} - e^0\right]$$

$$= \lim_{a\to\infty} -\frac{1}{s}\left[e^{-a\cdot s} - 1\right] = \begin{cases} \infty & \text{if } s \le 0 \\ \frac{1}{s} & \text{if } s > 0 \end{cases}$$

$$\boxed{\mathcal{L}[1] = \frac{1}{s},\ s > 0}$$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at} \cdot e^{-st} dt = \lim_{b\to\infty} \int_0^b e^{(a-s)t} dt$$

$$= \lim_{b\to\infty} \frac{1}{a-s} e^{(a-s)t} \Big|_0^b$$

$$= \frac{1}{a-s} \cdot \lim_{b\to\infty} \left[e^{(a-s)b} - e^0\right]$$

$$= \frac{1}{a-s} \begin{cases} \infty & \text{if } s \le a \\ -1 & \text{if } s > a \end{cases}$$

$$= \frac{-1}{a-s} \quad s > a$$

$$= \frac{1}{s-a} \quad s > a$$

$$\boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a},\ s > a}$$

Properties of the Laplace Transform

① Laplace transform of derivatives

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

② Linearity of the Laplace Transform

(a) $\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$

(b) $\mathcal{L}[cf] = c\mathcal{L}[f]$

Verify: $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty \frac{dy}{dt} \cdot e^{-st} \cdot dt$$

Integration by Parts

$$\boxed{\int u\,dv = u \cdot v - \int v\,du}$$

Let $u = e^{-st}$ $\quad \frac{du}{dt} = -se^{-st}$ $\quad du = -se^{-st} \cdot dt$

Let $dv = \frac{dy}{dt} \cdot dt \Rightarrow v = y(t)$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty e^{-st} \cdot \frac{dy}{dt} \cdot dt$$

$$= \lim_{b\to\infty} e^{-st} \cdot y(t) \Big|_{t=0}^{t=b} - \int_0^\infty -se^{-st} \cdot y(t)\, dt$$

$$= \lim_{b\to\infty} e^{-sb} \cdot y(b) - y(0) + s\int_0^\infty e^{-st} \cdot y(t)\, dt$$

$$= s\mathcal{L}[y] - y(0) + \lim_{b\to\infty} e^{-sb} \cdot y(b)$$

Assumption: $y(t)$ has at most exponential growth.

$\Rightarrow \quad |y(t)| < e^{Mt}$ for some constant M

$$\therefore \quad \lim_{t\to\infty} y(t)\cdot e^{-st} < \lim_{t\to\infty} e^{mt}\cdot e^{-st} = \lim_{t\to\infty} e^{(m-s)t} = 0$$

for sufficiently large $s > M$

$$\boxed{\therefore \quad \lim_{b\to\infty} e^{-sb}\cdot y(b) = 0}$$

Solving Differential Equations Using Laplace Transforms

Example: $\frac{dy}{dt} = y - 4e^{-t} \qquad y(0) = 1$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \mathcal{L}[y - 4e^{-t}]$$

$$s\,\mathcal{L}[y] - y(0) = \mathcal{L}[y] - 4\,\mathcal{L}[e^{-t}]$$

$$\boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a}$$

$$s\,\mathcal{L}[y] - y(0) = \mathcal{L}[y] - \frac{4}{s+1}$$

$$s\,\mathcal{L}[y] - \mathcal{L}[y] = 1 - \frac{4}{s+1}$$

$$(s-1)\,\mathcal{L}[y] = 1 - \frac{4}{s+1}$$

$$\mathcal{L}[y] = \frac{1}{s-1} - \frac{4}{(s-1)(s+1)}$$

$$\frac{dy}{dt} = y - 4e^{-t} \qquad y(0) = 1$$

Step 1 $\quad \frac{dy}{dt} - y = 0 \quad$ associated homogeneous eqn.

$$\frac{dy}{dt} = y \quad \Rightarrow \quad y_h(t) = Ke^{t}$$

Step 2 $\quad \frac{dy}{dt} - y = -4e^{-t} \qquad$ try $y_p(t) = Ae^{-t}$

$$y_p'(t) = -Ae^{-t}$$

$$-Ae^{-t} - Ae^{-t} = -4e^{-t}$$

$$-2A = -4 \quad \Rightarrow \quad A = +2$$

Step 3: General solution $y(t) = y_h(t) + y_p(t)$

$$y(t) = Ke^{t} + 2e^{-t}$$

$$y(0) = 1 = K + 2 \quad \Rightarrow \quad K = \text{[redacted]} -1$$

$$\boxed{\begin{array}{c} y(t) = \text{[redacted]}e^{t} + 2e^{-t} \\ -1e^{t} + 2e^{-t} \end{array}}$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}[F] = f \quad \text{if and only if} \quad \mathcal{L}[f] = F$$

Note: Uniqueness property of the Laplace Transform says: If f is a continuous function with $\mathcal{L}[f] = F$, then f is the only continuous function whose Laplace Transform is F.

Because the Laplace Transform is a linear operator $\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$ and $\mathcal{L}[cf] = c\,\mathcal{L}[f]$ for any functions $f(t)$, $g(t)$ and any constant c it follows that the inverse Laplace Transform is also linear.

$$\mathcal{L}^{-1}[F+G] = \mathcal{L}^{-1}[F] + \mathcal{L}^{-1}[G] \quad \text{and} \quad \mathcal{L}^{-1}[cF] = c\,\mathcal{L}^{-1}[F]$$

$$\mathcal{L}[y] = \frac{1}{s-1} - \frac{4}{(s-1)(s+1)}$$

$$\mathcal{L}^{-1}[\mathcal{L}[y]] = \mathcal{L}^{-1}\left[\frac{1}{s-1} - \frac{4}{(s-1)(s+1)}\right]$$

$$y = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{4}{(s-1)(s+1)}\right]$$

$$y(t) = e^{t} - \mathcal{L}^{-1}\left[\frac{a}{s-1} + \frac{b}{s+1}\right]$$

$$= e^{t} - \mathcal{L}^{-1}\left[\frac{2}{s-1} - \frac{2}{s+1}\right]$$

$$= e^{t} - 2\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$y(t) = e^{t} - 2e^{t} + 2e^{-t}$$

$$y(t) = -e^{t} - 2e^{-t}$$

$$-e^{t} + 2e^{-t}$$