Recall Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$$

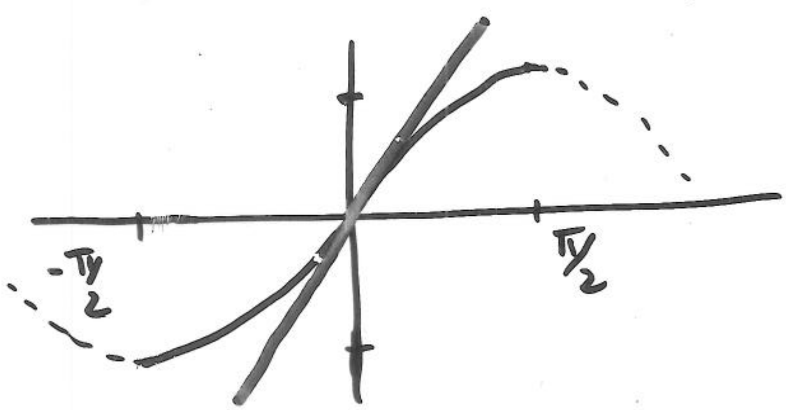
For example, $\sin x$ centered about $x=0$

$$\sin x = \sin(0) + \frac{\cos(0)}{1!}\cdot x - \frac{\sin(0)}{2!}x^2 \quad \frac{\cos(0)}{3!}x^3 - \cdots\cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$$

The Linearization of $\sin x$ about $x=0$ is

$$\sin x \approx x$$

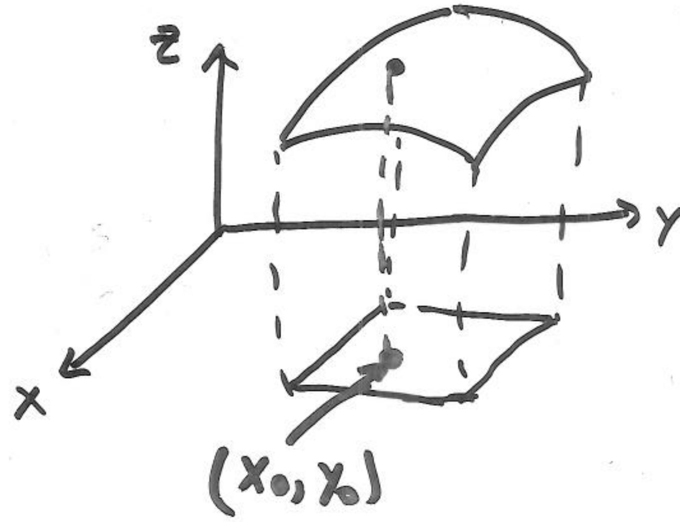


Claim: for "small" values of x, $-.1 < x < .1$

$$\sin x \approx x$$

In two dimensions

$$z = f(x,y)$$



we can approximate

z near (x_0, y_0)

as

$$z = f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x,y_0)(y-y_0)$$

Example: $z = f(x,y) = 2x^2 + y^2$ an elliptic paraboloid

Write the equation for the tangent plane at $(1,1,3)$

$$f_x(1,1) = 4x\Big|_{x=1} = 4$$

$$f_y(1,1) = 2y\Big|_{y=1} = 2$$

$$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$z = 3 + 4(x-1) + 2(y-1)$$

$$0 = 4(x-1) + 2(y-1) - (z-3)$$

plane passing through $(1,1,3)$
with Normal Vector
$(4,2,-1)$

Consider the Vander Pol Equation

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x + (1-x^2)y$$

Only one equilibrium point at $(0,0)$

What is the behavior of the equilibrium point?

Source, Sink, Saddle, Spiral Sink, Spiral Source, Center?

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x + y - x^2 y$$

Note for small values of (x,y) close to $(0,0)$ we can approximate the system as:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x + y$$

for $-.1 < x < .1$
$-.1 < y < .1$

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let x and y be two populations that compete for resources

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$

$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy$$

Equilibrium points: $x = 0$

then $\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right)$ $\quad y = 0$ or $y = 3$

$\Rightarrow (0,0)$ or $(0,3)$

If $y = 0$

then $\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right)$ $\quad x = 0,\ x = 2$

$\Rightarrow (0,0),\ (2,0)$

If $x \neq 0,\ y \neq 0$ ask

$$\begin{aligned} 2x\left(1 - \frac{x}{2}\right) - xy &= 0 \\ 3y\left(1 - \frac{y}{3}\right) - 2xy &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(2 - x - y) &= 0 \\ y(3 - y - 2x) &= 0 \end{aligned}$$

$$x(2-x-y)=0$$

$$y(3-y-2x)=0$$

If $x \neq 0$ and $y \neq 0$

$$\begin{aligned} 2-x-y &= 0 \\ 3-y-2x &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x+y &= 2 \quad ① \\ 2x+y &= 3 \quad ② \end{aligned}$$

Subtract ② - ① $\Rightarrow x=1$

$\Rightarrow y=1$

Four equilibrium points

$(0,0), (2,0), (0,3), (1,1)$

We want to study the system

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$

$$(*)$$

$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy$$

Near the equilibrium point $(1,1)$ using linearization.

<u>Note</u>: Linear systems only have equilibrium points at $(0,0)$

We need to modify the system so that the equilibrium point $(1,1)$ is moved to $(0,0)$

<u>change of variables</u>

Let $u = x - 1$, $v = y - 1$ $\Rightarrow$ $x = u+1$ $y = v+1$

$$\frac{du}{dt} = \frac{d(x-1)}{dt} = \frac{dx}{dt} \quad , \quad \frac{dv}{dt} = \frac{d(v-1)}{dt} = \frac{dy}{dt}$$

We rewrite the right hand side (r.h.s.) of $(*)$

$$\frac{dx}{dt} = 2(u+1)\left(1 - \frac{u+1}{2}\right) - (u+1)(v+1)$$

$$= 2u+2 - (u+1)^2 - uv - u - v - 1$$

$$= 2u + 2 - u^2 - 2u - 1 - uv - u - v - 1$$

$$= -u - v - u^2 - uv$$

Since $\frac{dx}{dt} = \frac{du}{dt}$ we have

$$\frac{du}{dt} = -u - v - u^2 - uv$$

<u>Next</u>:

$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy$$

$$\frac{dy}{dt} = 3y - y^2 - 2xy$$

$$\frac{dv}{dt} = 3(v+1) - (v+1)^2 - 2(u+1)(v+1)$$

$$\frac{dv}{dt} = 3(v+1) - \left(v^2 + 2v + 1\right) - 2\left(uv + u + v + 1\right)$$

$$= 3v + 3 - v^2 - 2v - 1 - 2uv - 2u - 2v - 2$$

$$\frac{dy}{dt} = -2u - v - 2uv - v^2$$

$$\frac{du}{dt} = -u - v - u^2 - uv$$

$$\frac{dv}{dt} = -2u - v - v^2 - 2uv$$

$$\frac{du}{dt} = -u - v$$

$$\frac{dv}{dt} = -2u - v$$

$$= \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

eigenvalues $\lambda = -1 \pm \sqrt{2}$

$\lambda_1 = -1 - \sqrt{2}$ $\lambda_2 = -1 + \sqrt{2}$

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -y - \sin x$$

Note $(0,0)$ is an equilibrium point

and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -y - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$$

For (x, y) close to $(0,0)$

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -y - x$$

We find eigenvalues $\lambda = (-1 \pm \sqrt{3}\, i)/2$

Linearization

General Form of a nonlinear system

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y)$$

Suppose (x_0, y_0) is an equilibrium point

We must first rewrite the system with (x_0, y_0) moved to $(0,0)$. Change of variables

Let $\quad u = x - x_0 \qquad \Rightarrow \qquad x = u + x_0$

$\quad\quad v = y - y_0 \qquad\qquad\quad y = v + y_0$

$$\frac{du}{dt} = \frac{d(x-x_0)}{dt} = \frac{dx}{dt} = f(x,y) = f(x_0+u,\ y_0+v)$$

$$\frac{dv}{dt} = \frac{d(y-y_0)}{dt} = \frac{dy}{dt} = g(x,y) = g(x_0+u,\ y_0+v)$$

$$\frac{du}{dt} = f(x_0+u,\ y_0+v)$$

$$\frac{dv}{dt} = g(x_0+u,\ y_0+v)$$

Recall $z = f(x,y)$

We approximate z near (x_0, y_0) as

$$z \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Let $u = x - x_0 \Rightarrow x = u + x_0$

$v = y - y_0 \Rightarrow y = v + y_0$

$z = f(x,y)$

$$z = f(u+x_0, v+y_0) \approx f(x_0,y_0) + f_x(x_0,y_0)u + f_y(x_0,y_0)v$$

$$\frac{du}{dt} = f(u+x_0, v+y_0) \cong f(x_0,y_0) + f_x(x_0,y_0)u + f_y(x_0,y_0)v$$

$$\frac{dv}{dt} = g(u+x_0, v+y_0) \cong g(x_0,y_0) + g_x(x_0,y_0)u + g_y(x_0,y_0)v$$

Note (x_0,y_0) is an equilibrium point

$\Rightarrow f(x_0,y_0) = 0 \qquad g(x_0,y_0) = 0$

$$\frac{du}{dt} = f_x(x_0,y_0)u + f_y(x_0,y_0)v$$

$$\frac{dv}{dt} = g_x(x_0,y_0)u + g_y(x_0,y_0)v$$

$$\frac{du}{dt} = f_x(x_0, y_0)u + f_y(x_0, y_0)v$$

$$\frac{dv}{dt} = g_x(x_0, y_0)u + g_y(x_0, y_0)v$$

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}}_{\text{Jacobian Matric}} \begin{bmatrix} u \\ v \end{bmatrix}$$

<u>Example</u>:

$$\frac{dx}{dt} = -2x + 2x^2$$

$$\frac{dy}{dt} = -3x + y + 3x^2$$

Two equilibrium points $(0,0)$, $(1,0)$

Compute the Jacobian Matrix

$$\begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix} = \begin{bmatrix} -2+4x & 0 \\ -3+6x & 1 \end{bmatrix}$$

at $(0,0)$ we have

$$\begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix}$$

$\lambda_1 = -2 \quad \lambda_2 = 1$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

at $(1,0)$

$$\begin{pmatrix} f_x(1,0) & f_y(1,0) \\ g_x(1,0) & g_y(1,0) \end{pmatrix} = \begin{pmatrix} +2 & 0 \\ +3 & 1 \end{pmatrix}$$

$\lambda_1 = 2 \quad \lambda_2 = 1$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$