Sec 4.3 Undamped Forcing + Resonance

$$\frac{d^2y}{dt^2} + 2y = \cos(\omega t)$$

Corresponding unforced equation

$$\frac{d^2y}{dt^2} + 2y = 0$$

has eigenvalues $s = \pm\sqrt{2}\, i$

$$y(t) = e^{\sqrt{2}ti} = \underbrace{\cos\sqrt{2}t}_{y_1(t)} + i\underbrace{\sin\sqrt{2}t}_{y_2(t)}$$

$$y_h(t) = k_1\cos\sqrt{2}t + k_2\sin\sqrt{2}t$$

No damping implies that solutions of the homogeneous equation oscillate for all time with period $T = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

and frequency $f = \frac{1}{T} = \frac{\sqrt{2}}{2\pi}$

$$\frac{d^2y}{dt^2} + ay = d\cdot\cos bt \qquad \begin{matrix} a>0 \\ b>0 \end{matrix}$$

Natural Response

$$\frac{d^2y}{dt^2} + ay = 0$$

$$s^2 + a = 0 \Rightarrow s = \pm\sqrt{-a}$$

$$s = \pm\sqrt{a}\, i$$

$$y(t) = e^{\sqrt{a}\,it}$$

$$y(t) = \underbrace{\cos\sqrt{a}\,t}_{y_1(t)} + i\underbrace{\sin\sqrt{a}\,t}_{y_2(t)}$$

Natural period $\frac{2\pi}{\sqrt{a}}$

Natural Frequency $\frac{\sqrt{a}}{2\pi}$

Forcing Fequency $\frac{b}{2\pi}$

$$\frac{d^2y}{dt^2} + 2y = \cos(\omega t) \qquad (*)$$

Find a particular solution, we will rewrite (*)

$$\frac{d^2y}{dt^2} + 2y = \qquad (**)$$

The real part of a solution of this equation is a solution of (*)

Try $y_p(t) = Ae^{i\omega t}$

$y_p'(t) = i\omega A e^{i\omega t}$

$y_p''(t) = -\omega^2 A e^{i\omega t}$

Plug into (**)

$$-\omega^2 A e^{i\omega t} + 2Ae^{i\omega t} = e^{i\omega t}$$

$$A(2-\omega^2)e^{i\omega t} = e^{i\omega t}$$

$$\Rightarrow A = \frac{1}{2-\omega^2}$$

$$Y_c(t) = \frac{1}{2-\omega^2} e^{i\omega t} = \frac{1}{2-\omega^2}\left(\cos\omega t + i\sin\omega t\right)$$

$$y_p(t) = \frac{1}{2-\omega^2}\cos(\omega t)$$

$$\frac{d^2y}{dt^2} + 2y = \cos\omega t$$

<u>General Solution</u>: $y(t) = y_h(t) + y_p(t)$

$$= k_1\cos\sqrt{2}t + k_2\sin\sqrt{2}t + \frac{1}{2-\omega^2}\cos\omega t$$

Consider initial conditions $y(0)=0$, $y'(0)=0$

$$y'(t) = -\sqrt{2}k_1\sin\sqrt{2}t + \sqrt{2}k_2\cos\sqrt{2}t - \frac{\omega}{2-\omega^2}\sin(\omega t)$$

$$y(0) = 0 = k_1 + \frac{1}{2-\omega^2} \Rightarrow k_1 = \frac{-1}{2-\omega^2}$$

$$y'(0) = 0 = \sqrt{2}k_2 \Rightarrow k_2 = 0$$

Solution of the initial value problem

$$y(t) = \frac{-1}{2-\omega^2}\cos\sqrt{2}t + \frac{1}{2-\omega^2}\cos\omega t$$

$$y(t) = \frac{1}{2-\omega^2}\left(\cos\omega t - \cos\sqrt{2}t\right)$$

(4)

Trig Identity

$$\cos x - \cos y = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$y(t) = \frac{1}{2-\omega^2}\left(\cos(\omega t) - \cos(\sqrt{2}t)\right)$$

$$= \frac{2}{2-\omega^2}\underbrace{\sin\left[\left(\frac{\omega+\sqrt{2}}{2}\right)t\right]}_{\text{fast}}\underbrace{\sin\left[\left(\frac{\omega-\sqrt{2}}{2}\right)t\right]}_{\text{slow}}$$

the beating period is $\dfrac{2\pi}{\frac{\omega-\sqrt{2}}{2}} = \dfrac{4\pi}{\omega-\sqrt{2}}$

the beating frequency $= \dfrac{1}{T} = \dfrac{\omega-\sqrt{2}}{4\pi}$

5)

The General Solution in the Resonant Case

$$\frac{d^2y}{dt^2} + 2y = \cos\sqrt{2}t \qquad (*)$$

$$y_h(t) = K_1\cos\sqrt{2}t + K_2\sin\sqrt{2}t$$

Try $y_p(t) = Ae^{i\sqrt{2}t}$

$y_p'(t) = i\sqrt{2}Ae^{i\sqrt{2}t}$

$y_p''(t) = -2Ae^{i\sqrt{2}t}$

plug-into $(*)$ $\quad -2Ae^{i\sqrt{2}t} + 2Ae^{i\sqrt{2}t} = e^{\sqrt{2}ti}$

Try $y_p(t) = Ate^{i\sqrt{2}t}$

$y_p'(t) = Ae^{i\sqrt{2}t} + i\sqrt{2}Ate^{i\sqrt{2}t}$

$y_p''(t) = i\sqrt{2}Ae^{i\sqrt{2}t} + i\sqrt{2}Ae^{i\sqrt{2}t} - 2At\cdot e^{i\sqrt{2}t}$

plug into $(*)$

$$2i\sqrt{2}Ae^{i\sqrt{2}t} - 2Ate^{i\sqrt{2}t} + 2Ate^{i\sqrt{2}t} = e^{i\sqrt{2}t}$$

$$2i\sqrt{2}A = 1$$

$$A = \frac{1}{2\sqrt{2}}\cdot\frac{1}{i} = \frac{-i}{2\sqrt{2}}$$

$$y_c(t) = \frac{-i}{2\sqrt{2}}\cdot te^{i\sqrt{2}t}$$

$$\frac{d^2y}{dt^2} + 2y = \cos\sqrt{2}t$$

$$Y_c(t) = \frac{-i}{2\sqrt{2}} \cdot t e^{i\sqrt{2}t}$$

$$Y_c(t) = \frac{-i}{2\sqrt{2}} t\left(\cos\sqrt{2}t + i\sin\sqrt{2}t\right)$$

$$y_c(t) = \frac{1}{2\sqrt{2}} t\sin(\sqrt{2}t) - \frac{i}{2\sqrt{2}} t\cos(\sqrt{2}t)$$

$$y_p(t) = Re(Y_c(t))$$

$$= \frac{1}{2\sqrt{2}} t\sin(\sqrt{2}t)$$