

Sec 4.3 Undamped Forcing + Resonance

(1)

$$\frac{d^2 y}{dt^2} + 2y = \cos(\omega t)$$

corresponding unforced equation

$$\frac{d^2 y}{dt^2} + 2y = 0$$

has eigenvalues $s = \pm \sqrt{2} i$

$$y(t) = e^{\sqrt{2}ti} = \underbrace{\cos \sqrt{2}t}_{y_1(t)} + i \underbrace{\sin \sqrt{2}t}_{y_2(t)}$$

$$y_h(t) = K_1 \cos \sqrt{2}t + K_2 \sin \sqrt{2}t$$

No damping implies that solutions of the homogeneous equation oscillate for all time

with period $T = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

and frequency $f = \frac{1}{T} = \frac{\sqrt{2}}{2\pi}$

$$\frac{d^2 y}{dt^2} + ay = d \cdot \cos bt \quad \begin{matrix} a > 0 \\ b > 0 \end{matrix}$$

Natural Response

$$\frac{d^2 y}{dt^2} + ay = 0$$

$$s^2 + a = 0 \quad \Rightarrow \quad \begin{matrix} s = \pm \sqrt{-a} \\ s = \pm \sqrt{a} i \end{matrix}$$

$$y(t) = e^{\sqrt{a} i t}$$

$$y(t) = \underbrace{\cos \sqrt{a} t}_{y_1(t)} + i \underbrace{\sin \sqrt{a} t}_{y_2(t)}$$

Natural period $\frac{2\pi}{\sqrt{a}}$

Natural Frequency $\frac{\sqrt{a}}{2\pi}$

Forcing Frequency $\frac{b}{2\pi}$

$$\frac{d^2 y}{dt^2} + 2y = \cos(\omega t) \quad (*)$$

Find a particular solution, we will rewrite (*)

$$\frac{d^2 y}{dt^2} + 2y = \quad (**)$$

The real part of a solution of this equation is a solution of (*)

$$\begin{aligned} \text{Try } y_p(t) &= A e^{i\omega t} \\ y_p'(t) &= i\omega A e^{i\omega t} \\ y_p''(t) &= -\omega^2 A e^{i\omega t} \end{aligned}$$

Plug into (**)

$$\begin{aligned} -\omega^2 A e^{i\omega t} + 2A e^{i\omega t} &= e^{i\omega t} \\ A(2 - \omega^2) e^{i\omega t} &= e^{i\omega t} \end{aligned}$$

$$\Rightarrow A = \frac{1}{2 - \omega^2}$$

$$y_c(t) = \frac{1}{2 - \omega^2} e^{i\omega t} = \frac{1}{2 - \omega^2} (\cos \omega t + i \sin \omega t)$$

$$y_p(t) = \frac{1}{2 - \omega^2} \cos(\omega t)$$

$$\frac{d^2 y}{dt^2} + 2y = \cos \omega t$$

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General Solution: $y(t) = y_h(t) + y_p(t)$

$$= K_1 \cos \sqrt{2}t + K_2 \sin \sqrt{2}t + \frac{1}{2-\omega^2} \cos \omega t$$

Consider initial conditions $y(0)=0$, $y'(0)=0$

$$y'(t) = -\sqrt{2}K_1 \sin \sqrt{2}t + \sqrt{2}K_2 \cos \sqrt{2}t - \frac{\omega}{2-\omega^2} \sin(\omega t)$$

$$y(0)=0 = K_1 + \frac{1}{2-\omega^2} \Rightarrow K_1 = \frac{-1}{2-\omega^2}$$

$$y'(0)=0 = \sqrt{2}K_2 \Rightarrow K_2 = 0$$

Solution of the initial value problem

$$y(t) = \frac{-1}{2-\omega^2} \cos \sqrt{2}t + \frac{1}{2-\omega^2} \cos \omega t$$

$$y(t) = \frac{1}{2-\omega^2} (\cos \omega t - \cos \sqrt{2}t)$$

Trig Identity

$$\cos x - \cos y = 2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

$$\begin{aligned} y(t) &= \frac{1}{2-\omega^2} (\cos(\omega t) - \cos(\sqrt{2}t)) \\ &= \frac{2}{2-\omega^2} \underbrace{\sin\left[\left(\frac{\omega+\sqrt{2}}{2}\right)t\right]}_{\text{fast}} \underbrace{\sin\left[\left(\frac{\omega-\sqrt{2}}{2}\right)t\right]}_{\text{slow}} \end{aligned}$$

the beating period is $\frac{2\pi}{\frac{\omega-\sqrt{2}}{2}} = \frac{4\pi}{\omega-\sqrt{2}}$

the beating frequency = $\frac{1}{T} = \frac{\omega-\sqrt{2}}{4\pi}$

5)

The General Solution in the Resonant Case

$$\frac{d^2 y}{dt^2} + 2y = \cos \sqrt{2} t \quad (*)$$

$$y_h(t) = K_1 \cos \sqrt{2} t + K_2 \sin \sqrt{2} t$$

$$\text{Try } y_p(t) = A e^{i\sqrt{2}t}$$

$$y_p'(t) = i\sqrt{2} A e^{i\sqrt{2}t}$$

$$y_p''(t) = -2 A e^{i\sqrt{2}t}$$

$$\text{plug into } (*) \quad -2 A e^{i\sqrt{2}t} + 2 A e^{i\sqrt{2}t} = e^{i\sqrt{2}t}$$

$$\text{Try } y_p(t) = A t e^{i\sqrt{2}t}$$

$$y_p'(t) = A e^{i\sqrt{2}t} + i\sqrt{2} A t e^{i\sqrt{2}t}$$

$$y_p''(t) = i\sqrt{2} A e^{i\sqrt{2}t} + i\sqrt{2} A e^{i\sqrt{2}t} - 2 A t e^{i\sqrt{2}t}$$

plug into (*)

$$2i\sqrt{2} A e^{i\sqrt{2}t} - 2 A t e^{i\sqrt{2}t} + 2 A t e^{i\sqrt{2}t} = e^{i\sqrt{2}t}$$

$$2i\sqrt{2} A = 1$$

$$A = \frac{1}{2\sqrt{2}} \cdot \frac{1}{i} = \frac{-i}{2\sqrt{2}}$$

$$y_c(t) = \frac{-i}{2\sqrt{2}} \cdot t e^{i\sqrt{2}t}$$

$$\frac{d^2 y}{dt^2} + 2y = \cos \sqrt{2} t$$

$$Y_c(t) = \frac{-i}{2\sqrt{2}} \cdot t e^{i\sqrt{2}t}$$

$$Y_c(t) = \frac{-i}{2\sqrt{2}} t (\cos \sqrt{2} t + i \sin \sqrt{2} t)$$

$$Y_c(t) = \frac{1}{2\sqrt{2}} t \sin(\sqrt{2} t) - \frac{i}{2\sqrt{2}} t \cos(\sqrt{2} t)$$

$$y_p(t) = \operatorname{Re}(Y_c(t))$$

$$= \frac{1}{2\sqrt{2}} t \sin(\sqrt{2} t)$$