Sec 4.1 Forced Harmonic Oscillators

Recall the damped harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$m>0,\ k>0,\ b\geq 0$

We introduce a forcing function $f(t)$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t)$$

← Forced Harmonic Oscillator

Dividing through by m

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{f(t)}{m}$$

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

linear, 2nd order, constant coefficient, nonhomogeneous differential equation.

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

associated homogeneous equation or unforced.

Extended Linearity Principle

Consider a non-homogeneous equation (a forced equation)

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t) \qquad (*)$$

and its associated homogeneous equation (the unforced eqn.)

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 \qquad (**)$$

1.) Suppose $y_p(t)$ is a particular solution of $(*)$ and $y_h(t)$ is a solution of $(**)$. Then $y_h(t) + y_p(t)$ is also a solution of $(*)$.

2.) Suppose $y_p(t)$ and $y_q(t)$ are both solutions of $(*)$. Then $y_q(t) - y_p(t)$ is a solution of $(**)$.

Therefore, if $k_1y_1(t) + k_2y_2(t)$ is the general solution of the associated homogeneous equation $(**)$ then the general solution of the nonhomogeneous equation $(*)$ is $k_1y_1(t) + k_2y_2(t) + y_p(t)$.

If $y_p(t)$ solves

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t) \qquad (*)$$

and $y_h(t)$ solves

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 \qquad (**)$$

then the function $y_p(t) + y_h(t)$ also solves $(*)$

Proof:

$$\frac{d^2(y_p(t)+y_h(t))}{dt^2} + p\frac{d(y_p(t)+y_h(t))}{dt} + q(y_p(t)+y_h(t)) =$$

$$\frac{d^2}{dt^2}y_p(t) + \frac{d^2}{dt^2}y_h(t) + p\frac{dy_p}{dt} + p\frac{dy_h}{dt} + qy_p(t) + qy_h(t) =$$

$$\frac{d^2y_p}{dt} + p\frac{dy_p}{dt} + qy_p + \frac{d^2}{dt}y_h + p\frac{d}{dt}y_h + qy_h =$$

If both $y_q(t)$ and $y_p(t)$ solve

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t) \qquad (*)$$

then $(y_q(t) - y_p(t))$ solves

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 \qquad (**)$$

Proof:

$$\frac{d^2}{dt^2}(y_q(t)-y_p(t)) + p\frac{d}{dt}(y_q(t)-y_p(t)) + q(y_q(t)-y_p(t)) =$$

$$\frac{d^2}{dt}y_q(t) - \frac{d^2}{dt}y_p(t) + p\frac{d}{dt}y_q(t) - p\frac{d}{dt}y_p(t) + qy_q(t) - qy_p(t) =$$

$$\frac{d^2}{dt}y_q(t) + p\frac{dy_q}{dt} + qy_q - \frac{d^2}{dt^2}y_p(t) - p\frac{dy_p}{dt} - qy_p =$$

Note: $Y_q(t) - Y_p(t) = Y_h(t)$

$Y_q(t) - Y_p(t) = K_1 Y_1(t) + K_2 Y_2(t)$

Example 1

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t}$$

$$s^2 + 5s + 6 = 0$$

$$(s+3)(s+2) = 0$$

$$s = -2, \quad s = -3$$

$$y_h(t) = K_1 e^{-2t} + K_2 e^{-3t}$$

$$y_p(t) =$$

$$y(t) = Y_h(t) + Y_p(t)$$

Example 2

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3e^{-2t}$$

Example 3

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 4e^{-3t}$$

Example 4

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 25y = 4t$$

Example 5

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\sin(2t)$$

Fact: If $y_p(t)$ solves

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

and $y_q(t)$ solves

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t)$$

then $y_q(t) + y_p(t)$ solves $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t) + f(t)$

Proof:

$$\frac{d^2}{dt^2}(y_q(t) + y_p(t)) + p\frac{d(y_q + y_p)}{dt} + q(y_p(t) + y_q(t)) =$$

$$\frac{d^2}{dt^2}y_q(t) + p\frac{dy_q}{dt} + qy_q + \frac{d^2}{dt^2}y_p(t) + p\frac{dy_p}{dt} + qy_p =$$

Example 6 :

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t} + \cos(2t) + 3t$$