

Sec 4.1 Forced Harmonic Oscillators

Recall the damped harmonic oscillator

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + k y = 0$$

$$m > 0, k > 0, b \geq 0$$

We introduce a forcing function $f(t)$

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + k y = f(t)$$

← Forced
Harmonic
Oscillator

Dividing through by m

$$\frac{d^2 y}{dt^2} + \frac{b}{m} \frac{dy}{dt} + \frac{k}{m} y = \frac{f(t)}{m}$$

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = g(t)$$

linear, 2nd order, constant coefficient,
non homogeneous differential equation.

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0$$

associated homogeneous equation or unforced.

Extended Linearity Principle

Consider a non-homogeneous equation (a forced equation)

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = g(t) \quad (*)$$

and its associated homogeneous equation (the unforced eqn.)

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0 \quad (**)$$

1.) Suppose $y_p(t)$ is a particular solution of (*) and $y_h(t)$ is a solution of (**). Then $y_h(t) + y_p(t)$ is also a solution of (*).

2.) Suppose $y_p(t)$ and $y_q(t)$ are both solutions of (*). Then $y_q(t) - y_p(t)$ is a solution of (**).

Therefore, if $k_1 y_1(t) + k_2 y_2(t)$ is the general solution of the associated homogeneous equation then the general solution of the nonhomogeneous equation (*) is $k_1 y_1(t) + k_2 y_2(t) + y_p(t)$.

If $y_p(t)$ solves

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = g(t) \quad (*)$$

and $y_h(t)$ solves

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0 \quad (**)$$

then the function $y_p(t) + y_h(t)$ also solves $(*)$

Proof:

$$\frac{d^2}{dt^2} (y_p(t) + y_h(t)) + p \frac{d}{dt} (y_p(t) + y_h(t)) + q (y_p(t) + y_h(t)) =$$

$$\frac{d^2}{dt^2} y_p(t) + \frac{d^2}{dt^2} y_h(t) + p \frac{dy_p}{dt} + p \frac{dy_h}{dt} + q y_p(t) + q y_h(t) =$$

$$\frac{d^2}{dt^2} y_p + p \frac{dy_p}{dt} + q y_p + \frac{d^2}{dt^2} y_h + p \frac{dy_h}{dt} + q y_h =$$

If both $y_q(t)$ and $y_p(t)$ solve

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = g(t) \quad (*)$$

then $(y_q(t) - y_p(t))$ solves

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0 \quad (**)$$

Proof:

$$\frac{d^2}{dt^2} (y_q(t) - y_p(t)) + p \frac{d}{dt} (y_q(t) - y_p(t)) + q (y_q(t) - y_p(t)) =$$

$$\frac{d^2}{dt^2} y_q(t) - \frac{d^2}{dt^2} y_p(t) + p \frac{dy_q}{dt} - p \frac{dy_p}{dt} + q y_q(t) - q y_p(t) =$$

$$\frac{d^2}{dt^2} y_q(t) + p \frac{dy_q}{dt} + q y_q - \frac{d^2}{dt^2} y_p(t) - p \frac{dy_p}{dt} - q y_p =$$

Note:

$$y_q(t) - y_p(t) = y_h(t)$$

$$y_q(t) - y_p(t) = k_1 y_1(t) + k_2 y_2(t)$$

Example 1

$$\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = e^{-t}$$

$$s^2 + 5s + 6 = 0$$

$$(s+3)(s+2) = 0$$

$$s = -2, s = -3$$

$$y_h(t) = k_1 e^{-2t} + k_2 e^{-3t}$$

$$y_p(t) =$$

$$y(t) = y_h(t) + y_p(t)$$

Example 2

$$\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = 3e^{-2t}$$

Example 3

$$\frac{d^2 y}{dt^2} + 6 \frac{dy}{dt} + 9y = 4e^{-3t}$$

Example 4

$$\frac{d^2 y}{dt^2} + 8 \frac{dy}{dt} + 25y = 4t$$

Example 5

$$\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = 3 \sin(2t)$$

Fact: If $y_p(t)$ solves

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = g(t)$$

and $y_q(t)$ solves

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = f(t)$$

then $y_q(t) + y_p(t)$ solves $\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = g(t) + f(t)$

Proof:

$$\begin{aligned} \frac{d^2}{dt^2} (y_q(t) + y_p(t)) + p \frac{d}{dt} (y_q + y_p) + q (y_p(t) + y_q(t)) &= \\ \frac{d^2}{dt^2} y_q(t) + p \frac{d}{dt} y_q + q y_q + q y_p + \frac{d^2}{dt^2} y_p + p \frac{d}{dt} y_p + q y_p &= \end{aligned}$$

Example 6 :

$$\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = e^{-t} + \cos(2t) + 3t$$