Sec 3.7 The Trace Determinant Plane

$$\frac{d\bar{y}}{dt} = A\bar{y} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

characteristic polynomial

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

Let $T = \text{trace} = a+d$

$D = \text{determinant} = ad - bc$

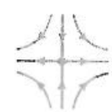
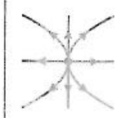
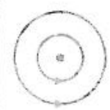
characteristic polynomial is rewritten as

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

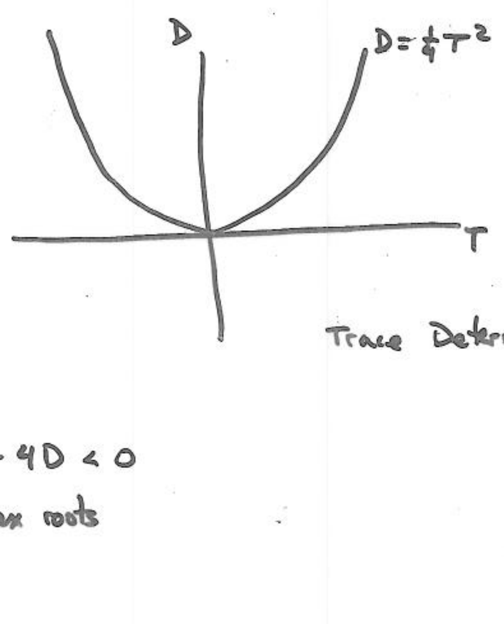
- If $T^2 - 4D < 0$ eigenvalues are complex
- If $T^2 - 4D = 0$ eigenvalues are repeated
- If $T^2 - 4D > 0$ eigenvalues are real + distinct

Table 3.1
Partial table of linear systems.

Type	Eigenvalues	Phase Plane	Type	Eigenvalues	Phase Plane
Saddle	$\lambda_1 < 0 < \lambda_2$		Spiral Sink	$\lambda = a \pm ib$, $a < 0, b \neq 0$	
Sink	$\lambda_1 < \lambda_2 < 0$		Spiral Source	$\lambda = a \pm ib$, $a > 0, b \neq 0$	
Source	$0 < \lambda_1 < \lambda_2$		Center	$\lambda = \pm ib$, $b \neq 0$	

$T^2 - 4D = 0 \Rightarrow D = \frac{1}{4}T^2$ parabola

peated
roots



Trace Determinant Plan

$T^2 - 4D < 0$
complex roots

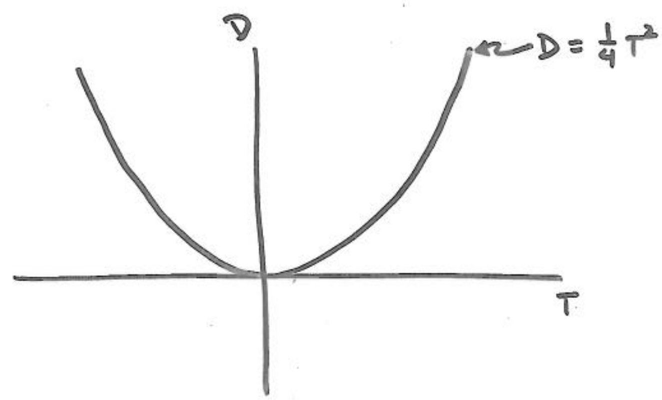
$T^2 - 4D > 0$
distinct real roots

Further refinements

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

Complex eigenvalues when $T^2 - 4D < 0$

or $D > \frac{1}{4}T^2$



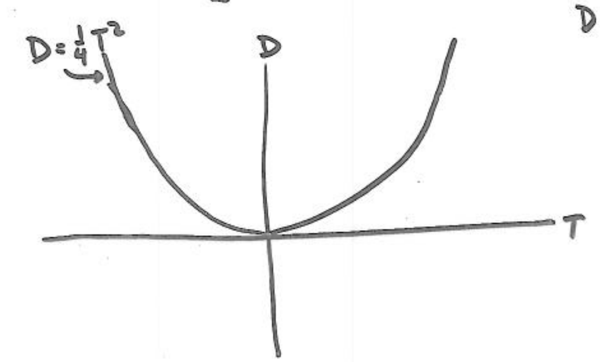
$T > 0$ equilibrium point at (0,0) is a spiral source

$T = 0$ equilibrium point at (0,0) is a center

$T < 0$ equilibrium point at (0,0) is a spiral sink

More refinements - real eigenvalues

$$\lambda = \frac{T \pm \sqrt{T^2-4D}}{2} \qquad T^2-4D>0 \quad \text{or} \quad D < \tfrac{1}{4}T^2$$



Let $T>0$

$$\lambda_1 = \frac{T+\sqrt{T^2-4D}}{2} > 0$$

$$\lambda_2 = \frac{T-\sqrt{T^2-4D}}{2}$$

Case 1: $D=0 \Rightarrow \lambda_2 = \frac{T-T}{2} = 0$

Case 2: $D>0 \Rightarrow T^2-4D < T^2 \Rightarrow \sqrt{T^2-4D} < T$

$$\therefore \lambda_2 = \frac{T-\sqrt{T^2-4D}}{2} > 0$$

Case 3: $D<0$ and $T>0$

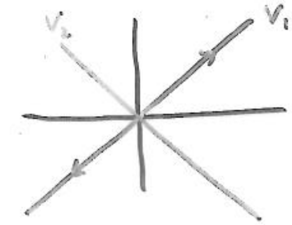
$$T^2-4D > T^2$$

$$\sqrt{T^2-4D} > T$$

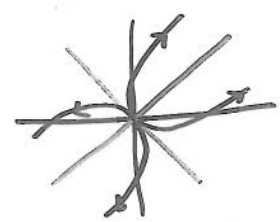
$$\therefore \lambda_2 = \frac{T-\sqrt{T^2-4D}}{2} < 0$$

Summary $T^2-4D>0$

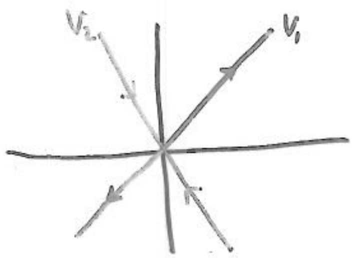
Case 1: $T>0$, $D=0$ $\qquad \lambda_1>0$, $\lambda_2=0$



Case 2: $T>0$, $D>0$ $\qquad \lambda_1>0$, $\lambda_2>0$



Case 3: $T>0$, $D<0$ $\qquad \lambda_1>0$, $\lambda_2<0$



Using the same reasoning for $T^2-4D>0$

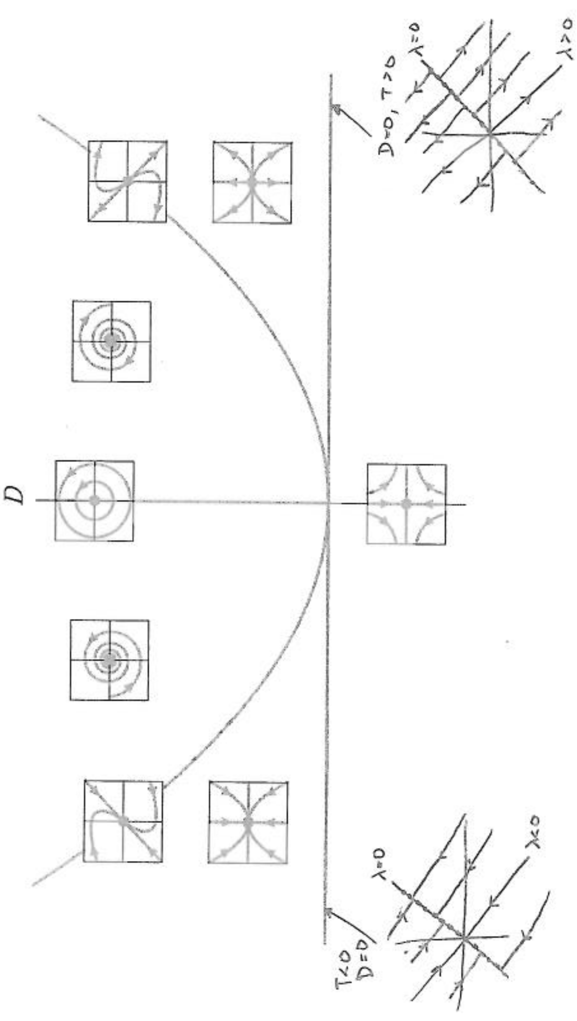
but $T<0$ we have

$$\lambda_1 = \frac{T-\sqrt{T^2-4D}}{2} < 0$$

but λ_2 depends on D

- $D=0 \qquad \lambda_2 = \frac{T+\sqrt{T^2}}{2} = 0$
- $D>0 \qquad \lambda_2 = \frac{T+\sqrt{T^2-4D}}{2} < 0$
- $D<0 \qquad \lambda_2 = \frac{T+\sqrt{T^2-4D}}{2} > 0$

The Big Picture



The Harmonic Oscillator in the Trace-Determinant Plane

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$$m, k > 0, \quad b \geq 0$$

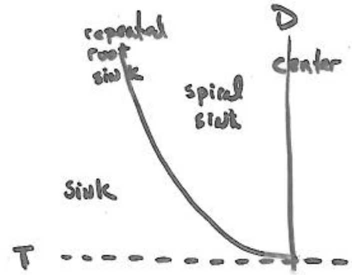
$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m}y - \frac{b}{m}v$$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}\bar{y}$$

Trace $T = -\frac{b}{m} \leq 0$ Determinant $D = \frac{k}{m} > 0$



Example: $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$

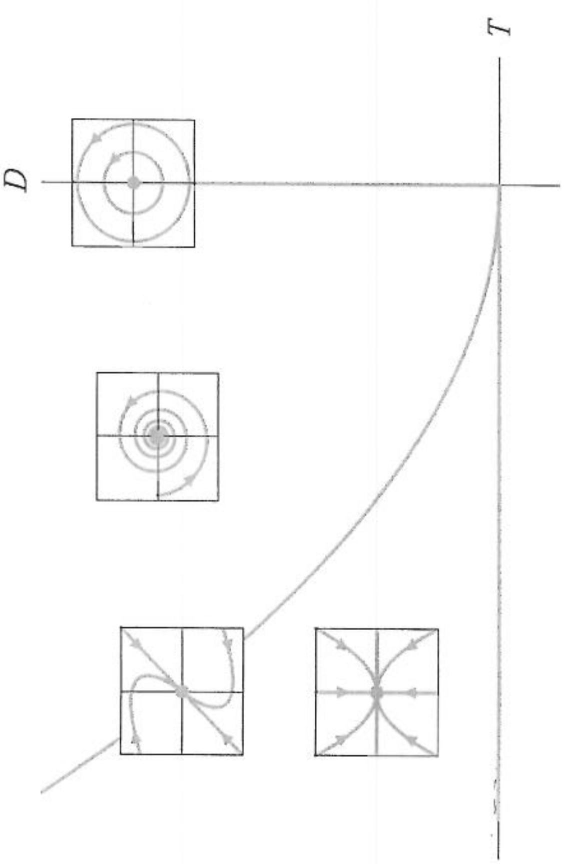
$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix}\bar{y}$$

$T = -7$ $D = 10$

$D < \frac{1}{4}T^2$

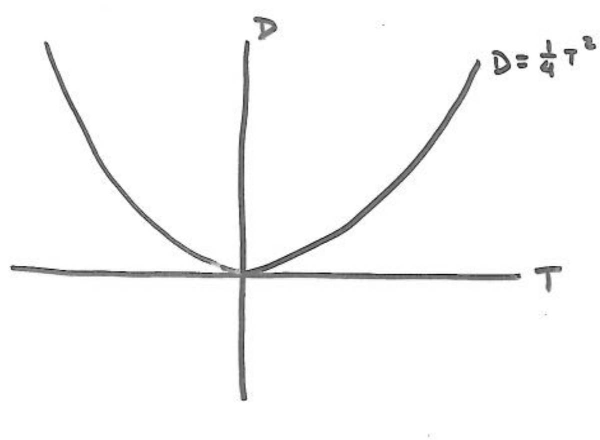
$10 < \frac{1}{4}\cdot 49 = 12.25$

Example: $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0$



Bifurcations in a one parameter family of Linear Systems

$\frac{d\bar{y}}{dt} = A\bar{y}$ $A = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}$ $T = -2$, $D = 2a$



$$\frac{d\bar{y}}{dt} = A\bar{y}$$

$$A = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}$$

$T = -2$

$D = 2a$

Let $-\infty < a < \infty$