

Sec 3.7 The Trace Determinant Plane

$$\frac{d\bar{y}}{dt} = A\bar{y} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

characteristic polynomial

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

Let $T = \text{trace} = a+d$

$D = \text{determinant} = ad-bc$

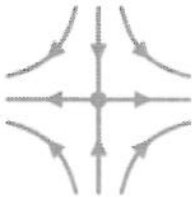
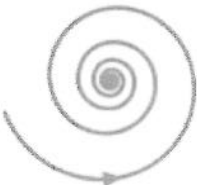
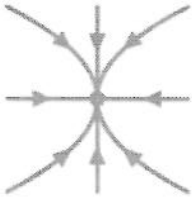
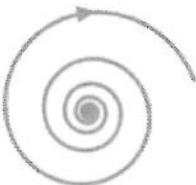
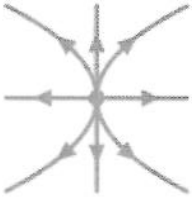
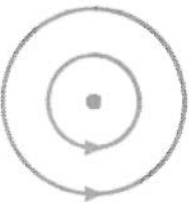
characteristic polynomial is rewritten as

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

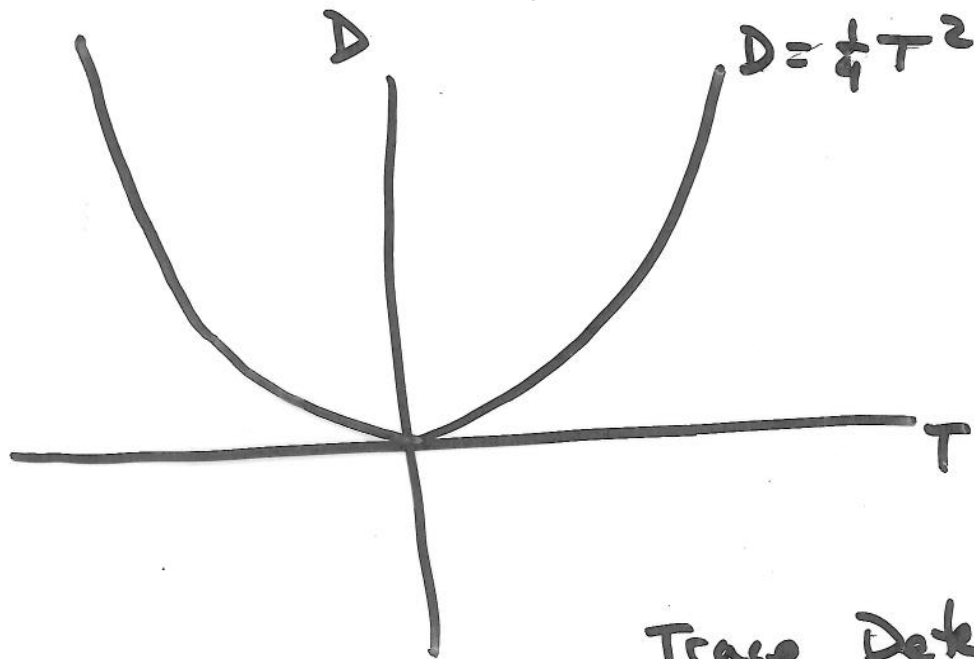
- If $T^2 - 4D < 0$ eigenvalues are complex
- If $T^2 - 4D = 0$ eigenvalues are repeated
- If $T^2 - 4D > 0$ eigenvalues are real + distinct

Table 3.1
Partial table of linear systems.

Type	Eigenvalues	Phase Plane	Type	Eigenvalues	Phase Plane
Saddle	$\lambda_1 < 0 < \lambda_2$		Spiral Sink	$\lambda = a \pm ib$ $a < 0, b \neq 0$	
Sink	$\lambda_1 < \lambda_2 < 0$		Spiral Source	$\lambda = a \pm ib$ $a > 0, b \neq 0$	
Source	$0 < \lambda_1 < \lambda_2$		Center	$\lambda = \pm ib$ $b \neq 0$	

$$T^2 - 4D = 0 \Rightarrow D = \frac{1}{4}T^2 \text{ parabola}$$

repeated
roots



Trace Determinant Plane

$$T^2 - 4D < 0$$

complex roots

$$T^2 - 4D > 0$$

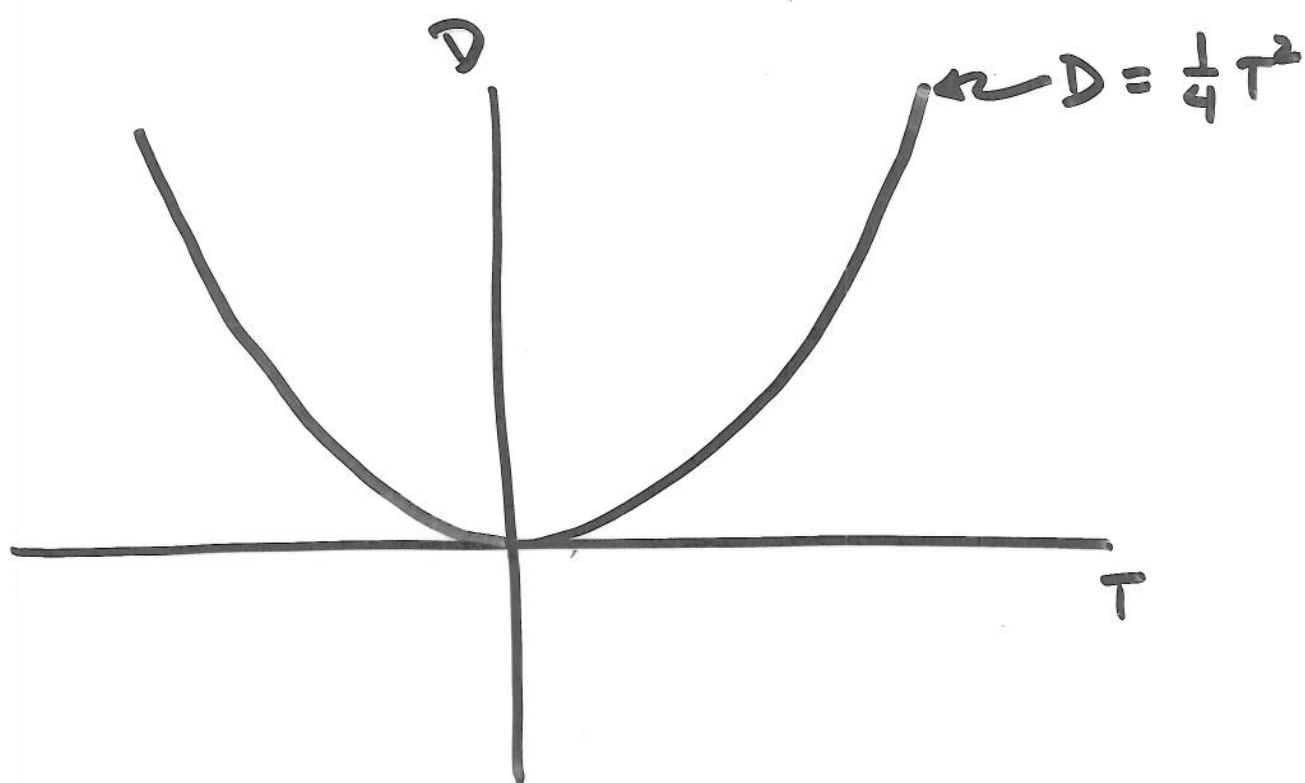
distinct real roots

Further refinements

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

Complex eigenvalues when $T^2 - 4D < 0$

$$\text{or } D > \frac{1}{4}T^2$$



$T > 0$ equilibrium point at $(0,0)$ is a spiral source

$T = 0$ equilibrium point at $(0,0)$ is a center

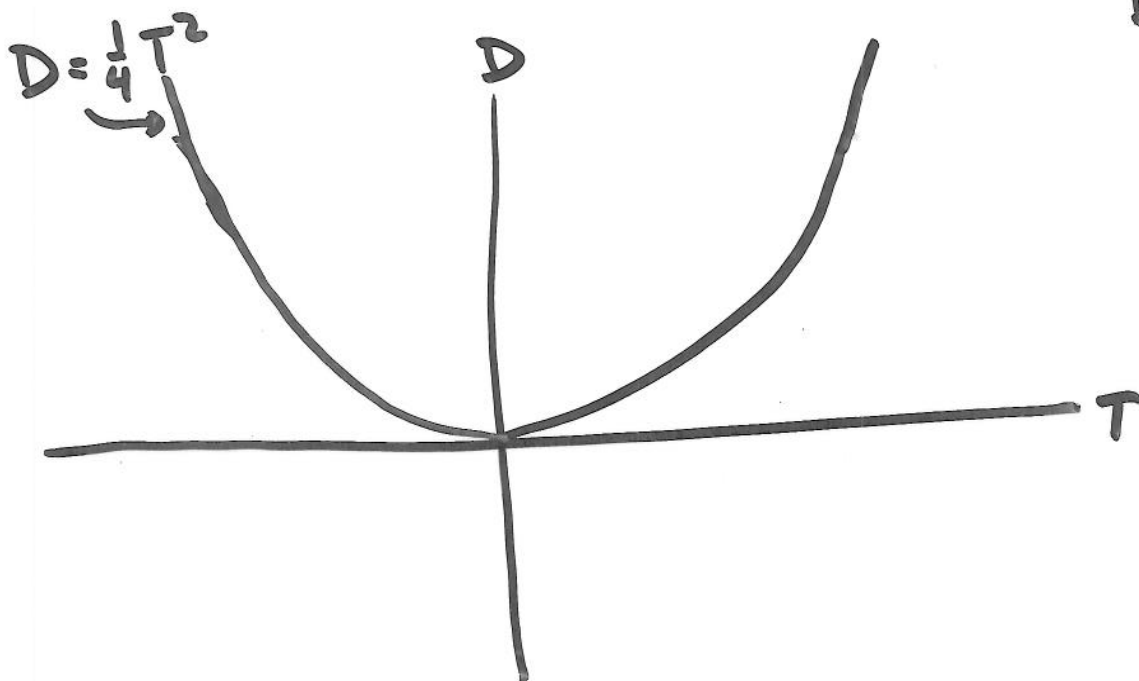
$T < 0$ equilibrium point at $(0,0)$ is a spiral sink

More refinements - real eigenvalues

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

$$T^2 - 4D > 0 \quad \text{or}$$

$$D < \frac{1}{4}T^2$$



Let $T > 0$

$$\lambda_1 = \frac{T + \sqrt{T^2 - 4D}}{2} > 0$$

$$\lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2}$$

Case 1: $D = 0 \Rightarrow \lambda_2 = \frac{T - T}{2} = 0$

Case 2: $D > 0 \Rightarrow T^2 - 4D < T^2 \Rightarrow \sqrt{T^2 - 4D} < T$

$$\therefore \lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2} > 0$$

Case 3: $D < 0$ and $T > 0$

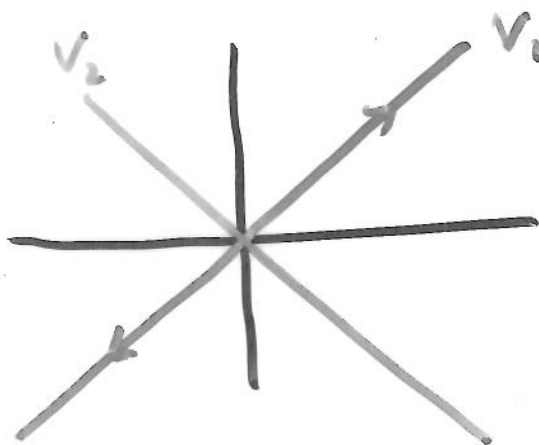
$$T^2 - 4D > T^2$$

$$\sqrt{T^2 - 4D} > T$$

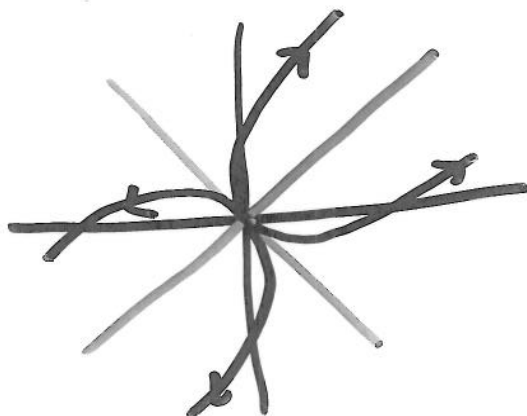
$$\therefore \lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2} < 0$$

Summary $T^2 - 4D > 0$

Case 1: $T > 0, D = 0 \quad \lambda_1 > 0, \lambda_2 = 0$

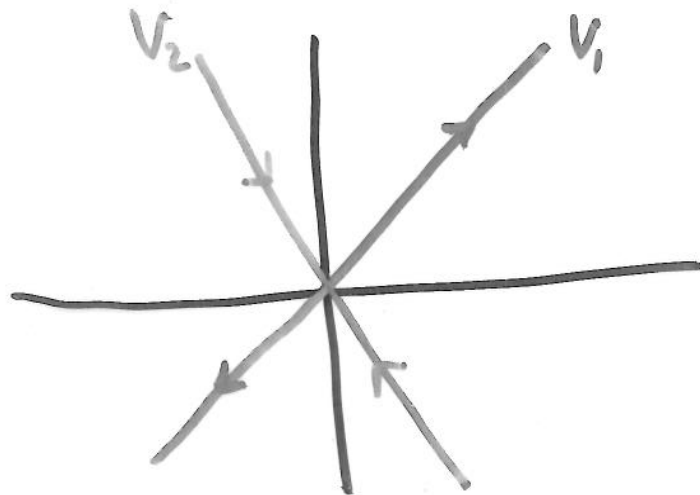


Case 2: $T > 0, D > 0 \quad \lambda_1 > 0, \lambda_2 > 0$



Case 3: $T > 0$, $D < 0$

$\lambda_1 > 0$, $\lambda_2 < 0$



using the same reasoning for $T^2 - 4D > 0$

but $T < 0$ we have

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < 0$$

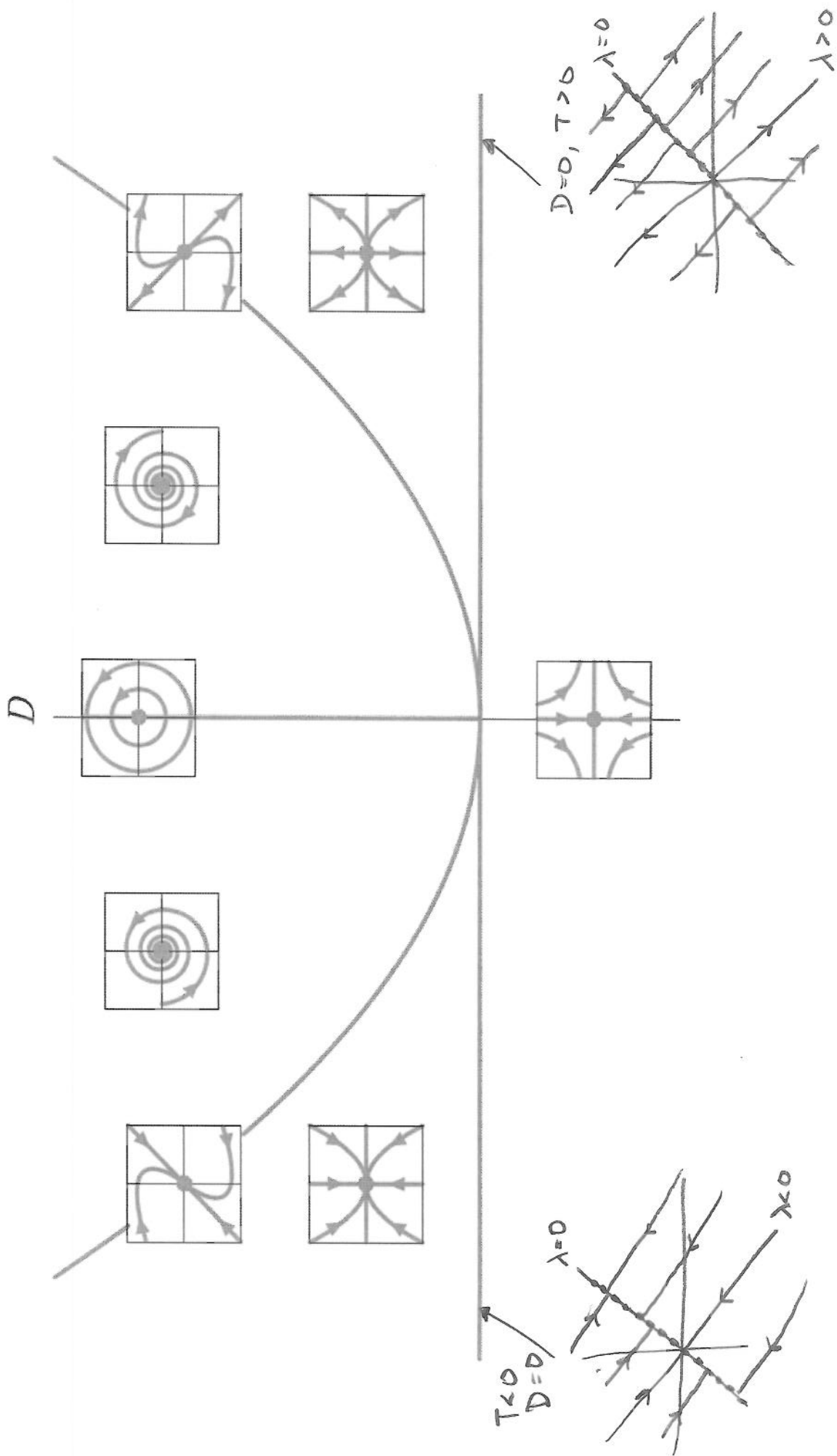
but λ_2 depends on D

• $D = 0$ $\lambda_2 = \frac{T + \sqrt{T^2}}{2} = 0$

• $D > 0$ $\lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2} < 0$

• $D < 0$ $\lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2} > 0$

The Big Picture



• The Harmonic Oscillator in the Trace-Determinant Plane

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0$$

$$m, k > 0, \quad b \geq 0$$

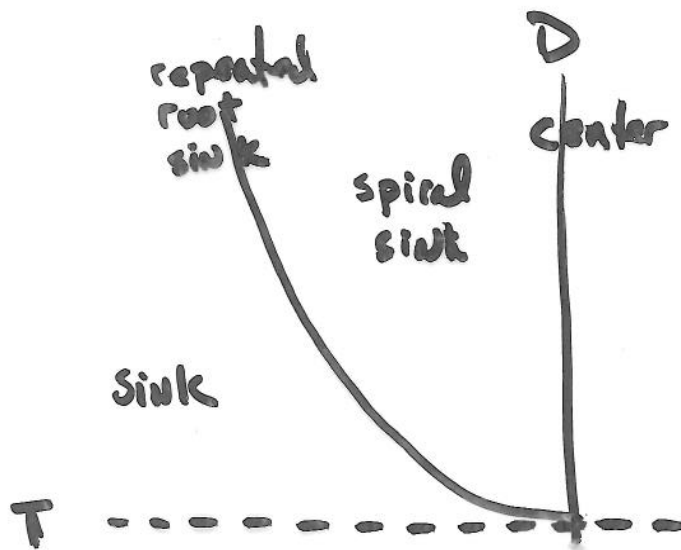
$$\frac{d^2 y}{dt^2} + \frac{b}{m} \frac{dy}{dt} + \frac{k}{m} y = 0$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m} y - \frac{b}{m} v$$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \bar{y}$$

$$\text{Trace } T = -\frac{b}{m} \leq 0 \quad \text{Determinant } D = \frac{k}{m} > 0$$



Example : $\frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 10y = 0$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix} \bar{y}$$

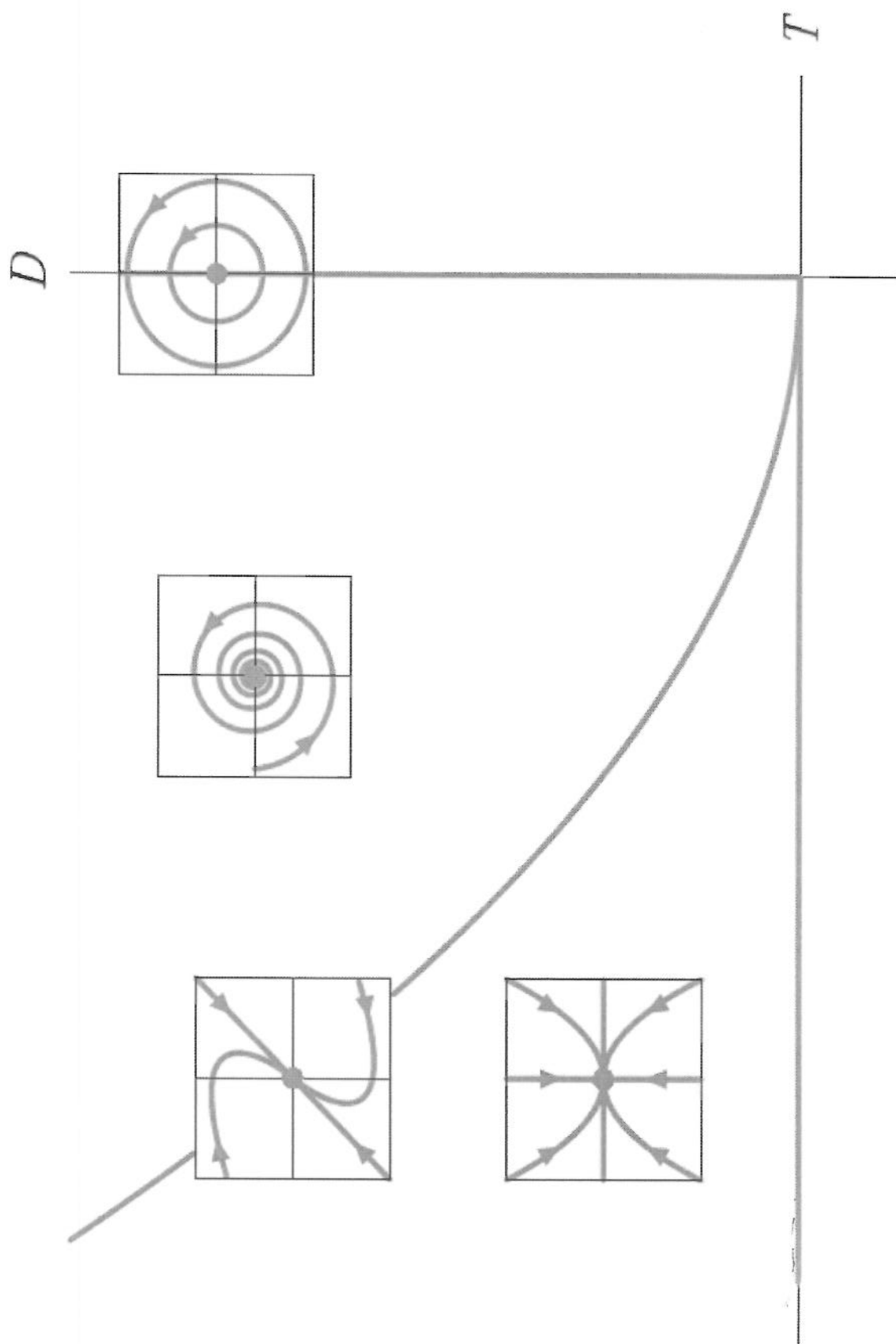
$$T = -7 \quad D = 10$$

$$D < \frac{1}{4} T^2$$

$$10 < \frac{1}{4} \cdot 49 = 12.25$$

Example : $\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 13y = 0$

The Harmonic Oscillator

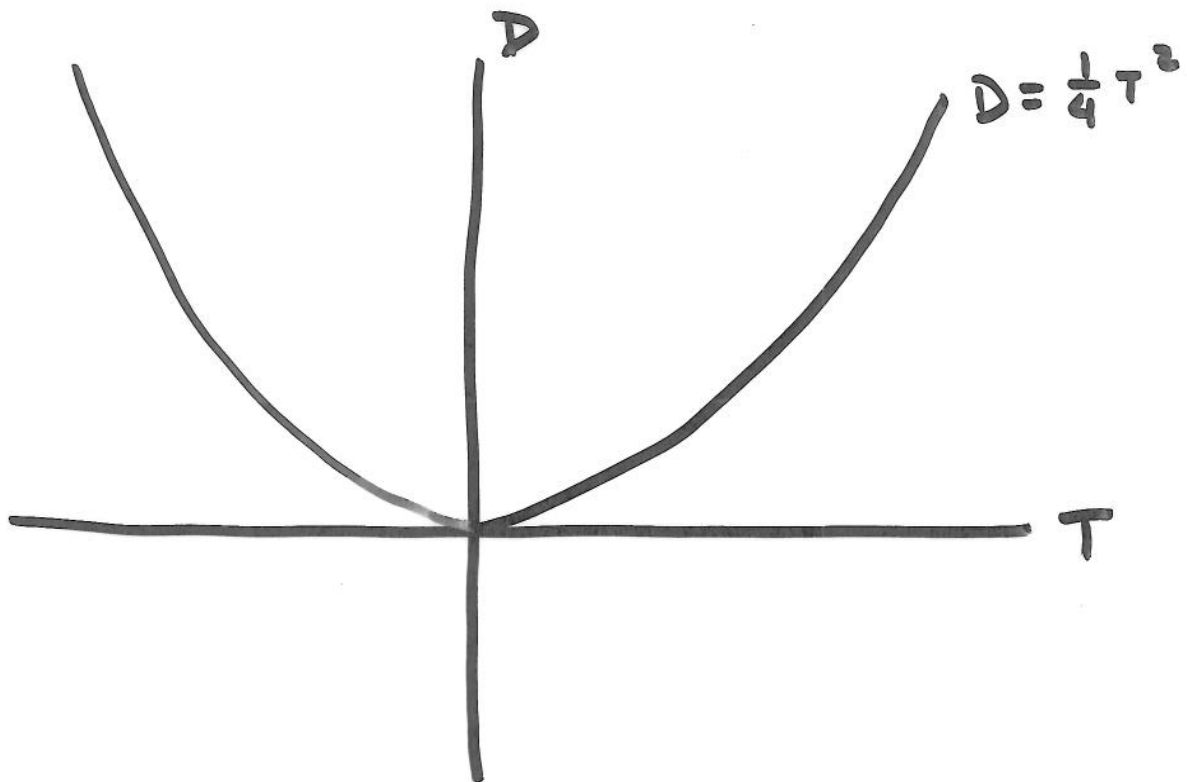


Bifurcations in a one parameter family of Linear Systems

$$\frac{d\bar{y}}{dt} = A\bar{y}$$

$$A = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}$$

$$T = -2$$
$$D = 2a$$



$$\frac{d\bar{y}}{dt} = A\bar{y}$$

$$A = \begin{pmatrix} -2 & \alpha \\ -2 & 0 \end{pmatrix}$$

$$T = -2$$

$$D = 2\alpha$$

$$\text{Let } -\infty < \alpha < \infty$$

