Sec 3.6 Second Order Linear Systems

Second order linear equations vs First-Order Systems

<u>Every</u> 2^{nd} order linear equation can be written as a first order system.

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \qquad m, b, k > 0$$

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 \qquad p = \frac{b}{m}, \; q = \frac{k}{m}$$

equivalent system

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\bar{y} \qquad \bar{y} = \begin{pmatrix} y \\ v \end{pmatrix}$$

<u>Note</u>: It is much easier to solve the 2^{nd} order differential equation than the first order system

<u>Example 1</u>: $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$ | $\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix}\bar{y}$

We try solutions of the form $y = e^{st}$

and plug into

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$

<u>Example 2</u>: $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -13 & -4 \end{pmatrix} \bar{y}$$

try $y(t) = e^{st}$

A Classification of Harmonic Oscillators

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \qquad m, b, k > 0$$

characteristic equation $\quad ms^2 + bs + k = 0$

$$s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2}$$

Cases

1.) $b = 0$, no damping. Roots are imaginary numbers
The equilibrium point at $(0,0)$ is a center.
The solutions are periodic, $T = 2\pi\sqrt{\frac{m}{k}}$

2.) $b > 0$ and $b^2 - 4mk > 0$, the oscillator is <u>overdamped</u>
The origin in the phase plane is a sink with 2 distinct eigenvalues. The mass tends to rest position but does <u>not</u> oscillate.

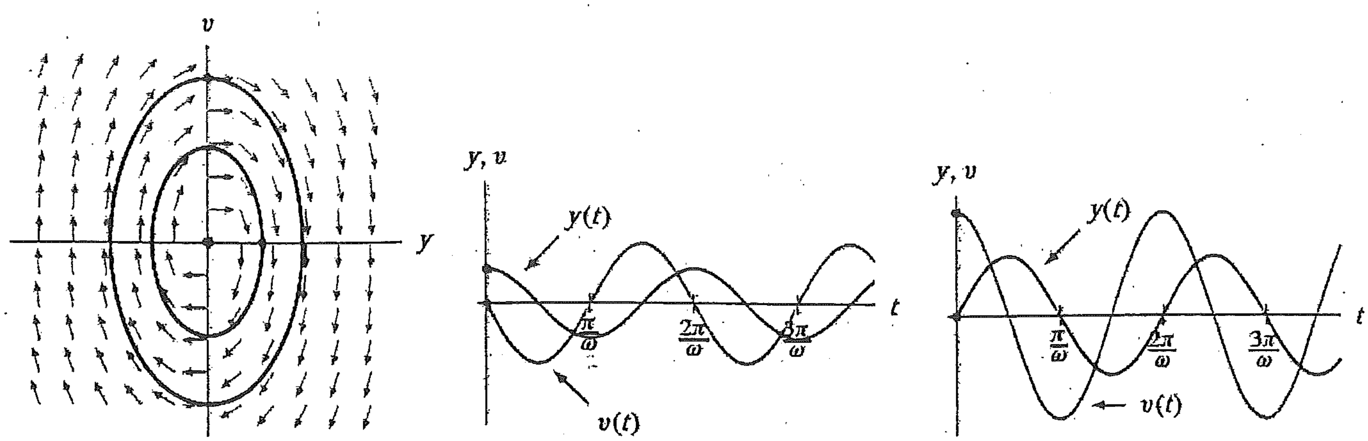
3.) $b > 0$ and $b^2 - 4mk < 0$, the oscillator is <u>underdamped</u>
The origin in the phase plane is a spiral sink.
The mass oscillates as it tends to its rest position with period $T = \dfrac{4m\pi}{\sqrt{4km - b^2}}$

Classificaton of Harmonic Oscillators

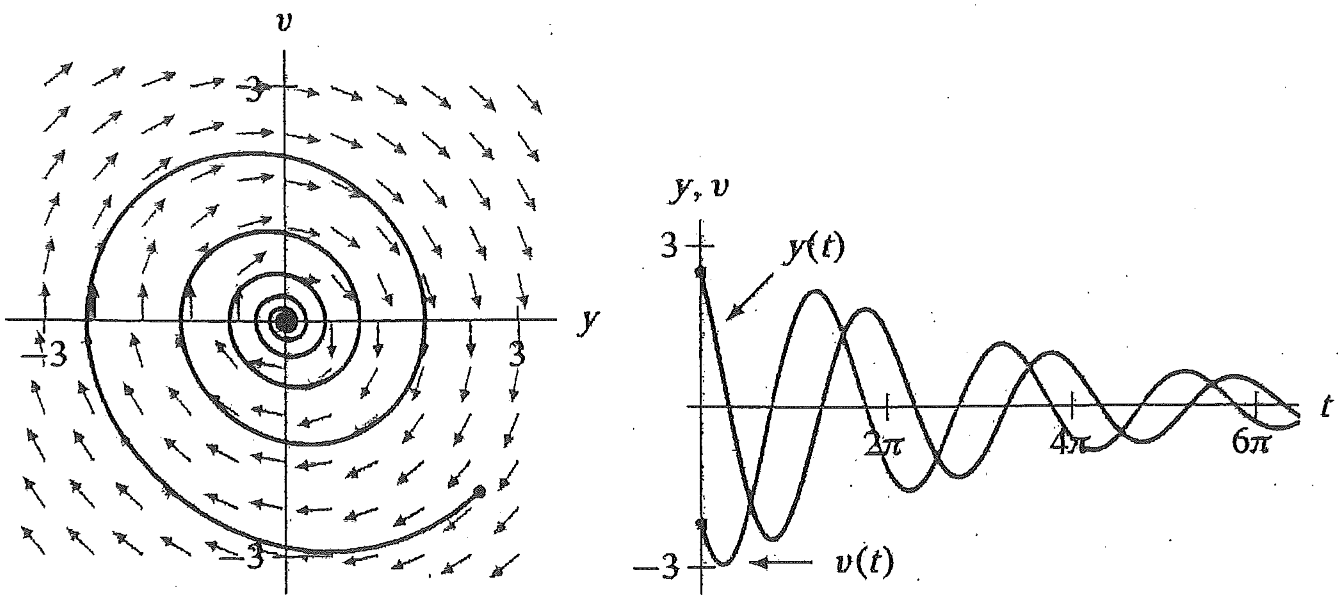
4.) If $b>0$ and $b^2-4mk=0$, the oscillator is critically damped. The system has one eigenvalue that is negative. All solutions tend to the origin in the phaseplane tangent to the unique line of eigenvectors.

The mass does oscillate.

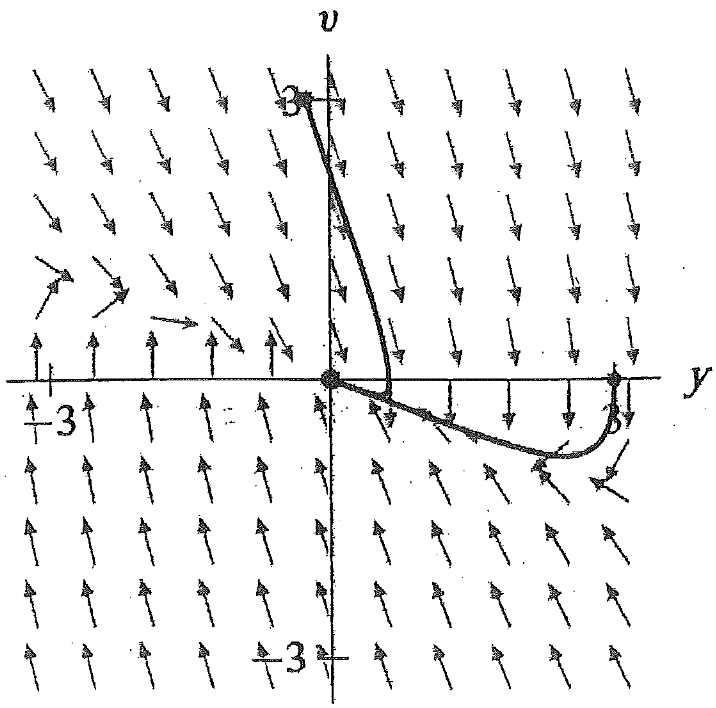
Undamped oscillator



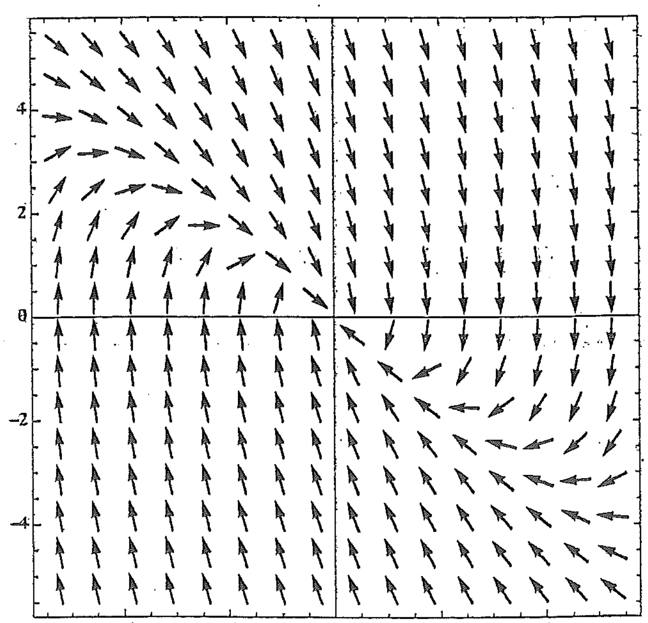
Underdamped oscillator



Overdamped Oscillator



Critically Damped Oscillator



Case 1 : The undamped harmonic oscillator

$$m\frac{d^2y}{dt^2} + ky = 0 \qquad b=0$$

characteristic polynomial $\quad ms^2 + k = 0$

$$s = \pm\sqrt{\frac{k}{m}}\, i$$

$$y(t) = e^{i\sqrt{\frac{k}{m}}t} = \cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + i\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$$y(t) = K_1\cos(\omega t) + K_2\sin(\omega t) \qquad \omega = \sqrt{\frac{k}{m}}$$

$$v(t) = -\omega K_1\sin(\omega t) + \omega K_2\cos(\omega t)$$

$$\bar{Y}(t) = K_1\begin{pmatrix}\cos(\omega t)\\ -\omega\sin(\omega t)\end{pmatrix} + K_2\begin{pmatrix}\sin(\omega t)\\ \omega\cos(\omega t)\end{pmatrix}$$

Solution is an ellipse in the phase plane

period $T = 2\pi/\omega = 2\pi\sqrt{\frac{m}{k}}$

Case 2 - The Underdamped Harmonic Oscillator

$$\frac{d^2y}{dt^2} + .2\frac{dy}{dt} + 1.01y = 0$$

characteristic polynomial $s^2 + .2s + 1.01 = 0$

$$s = -.1 + i \quad \text{or} \quad s = -.1 - i$$

$$y(t) = e^{st} = e^{-.1t} \cdot e^{it} = e^{-.1t}(\cos t + i \sin t)$$

<u>General Solution</u> $y(t) = k_1 e^{-.1t} \cos t + k_2 e^{-.1t} \sin t$

The solution has natural period of 2π but the amplitude decays as $t \to \infty$

In the phase plane we have

$$\bar{Y}(t) = k_1 e^{-.1t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + k_2 e^{-.1t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

Case 4 : Critically Damped

$$\frac{d^2y}{dt^2} + 2\sqrt{2}\frac{dy}{dt} + 2y = 0$$

$$s^2 + 2\sqrt{2}s + 2 = 0$$

repeated root $\quad s = -\sqrt{2}$

General solution

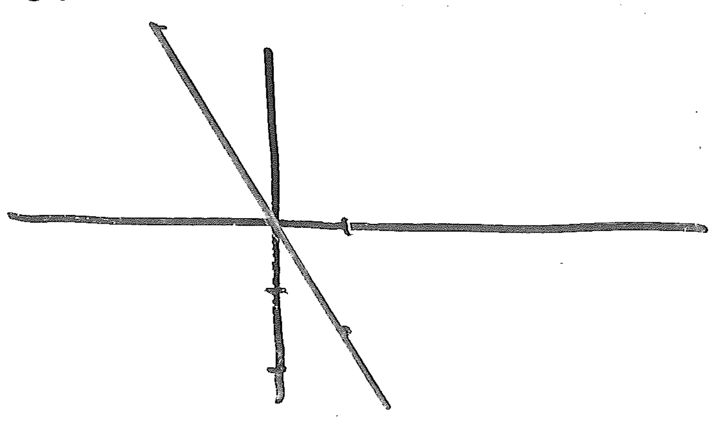
$$y(t) = k_1 e^{-\sqrt{2}t} + k_2 t e^{-\sqrt{2}t}$$

Because $s = -\sqrt{2} < 0$ all solutions tend to zero without oscillation just as in the overdamped case.

To see how the solutions tend to the origin it is best to look at the phase plane.

The system is $\displaystyle \frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{pmatrix}\bar{y}$

the eigenvector for $\lambda = -2$ is $\bar{v} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$



Case 3: Overdamped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$$

characteristic polynomial $s^2 + 3s + 1 = 0$

$$s = \frac{-3 \pm \sqrt{5}}{2} \qquad s_1 = \frac{-3+\sqrt{5}}{2} < 0 \,, \quad s_2 = \frac{-3-\sqrt{5}}{2} < 0$$

The general solution $y(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$

Since $s_1, s_2 < 0$ all solutions tend to the rest position as time goes forward.

To answer the question of how the solutions tend to rest we need the phase plane and eigenvectors.

Let $\bar{v}_1$ and $\bar{v}_2$ be the corresponding eigenvectors.

Since $s_2 < s_1 < 0$ we know that all solutions in the plane (except thoes on the line determined by v_2) tend to the origin tangent to the $\bar{v}_1$ direction