Sec 3.5 Special Cases: Repeated + zero Eigenvalues

$$\frac{d\bar{y}}{dt} = A\bar{y} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\bar{y}$$

$\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} -2-\lambda & 1 \\ 0 & (-2-\lambda) \end{pmatrix} = 0$$

$(-2-\lambda)(-2-\lambda) = 0$

$(\lambda+2)^2 = 0 \qquad \boxed{\lambda = -2}$

Find corresponding eigenvector

$A\bar{v} = -2\bar{v}$

$$\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left.\begin{aligned} -2x + y &= -2x \\ -2y &= -2y \end{aligned}\right\} \quad y = 0$$

eigenvector $\begin{pmatrix} x \\ 0 \end{pmatrix}$ or $x\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Another approach

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\bar{y} \iff \left.\begin{aligned} \frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= -2y \end{aligned}\right\} \text{partially decoupled}$$

$$\frac{dy}{dt} = y_0 e^{-2t}$$

(*) $\frac{dx}{dt} = -2x + y_0 e^{-2t}$

step 1 $\quad x_h(t) = x_0 e^{-2t} \qquad$ solves $\frac{dx}{dt} = -2x$

step 2 $\quad x_p(t) = Ate^{-2t}$

plug into $\quad Ae^{-2t} - 2Ate^{-2t} = -2Ate^{-2t} + y_0 e^{-2t}$

$\Rightarrow A = y_0$

$x(t) = x_h(t) + x_p(t) = x_0 e^{-2t} + y_0 te^{-2t}$

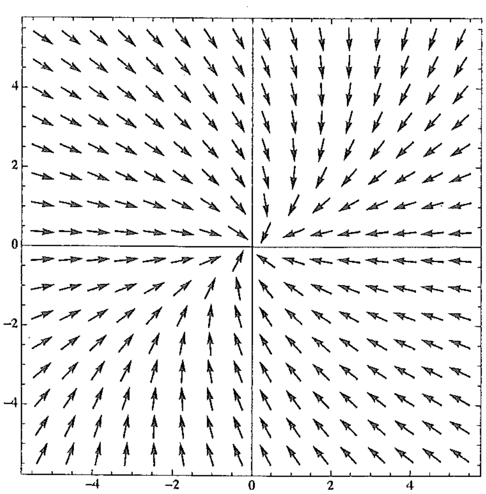
$$\bar{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{-2t} + y_0 t e^{-2t} \\ y_0 e^{-2t} \end{pmatrix}$$

$$\bar{y}(t) = e^{-2t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-2t}\begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\bar{y} = A\bar{y}$$

repeated eigenvalue $\lambda = -2$

eigenvector $\bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Motivated by this example we try something of the same form.

If $\frac{d\bar{y}}{dt} = A\bar{y}$ is a system with repeated eigenvalue λ

Try a solution of the form

$$\bar{y}(t) = e^{\lambda t}\bar{V}_0 + te^{\lambda t}\bar{V}_1$$

where $\bar{V}_0 = \bar{y}(0)$, the initial condition for $\bar{y}(t)$

We plug into the equation $\frac{d\bar{y}}{dt} = A\bar{y}$

$\frac{d\bar{y}}{dt} =$

$A\bar{y} =$

equating the left and right hand side

Theorem: Suppose $\frac{d\bar{Y}}{dt} = A\bar{Y}$ is a linear system in which the 2×2 matrix A has a real repeated eigenvalue λ and only one line of eigenvectors. Then the general solution has the form

$$\bar{Y}(t) = e^{\lambda t}\bar{V}_0 + te^{\lambda t}\bar{V}_1$$

where $\bar{V}_0 = (x_0, y_0)$ is an arbitrary initial condition and $\bar{V}_1 = (A - \lambda I)\bar{V}_0$.

If $\bar{V}_1 = \bar{0}$ then $\bar{V}_0$ is an eigenvector and $\bar{Y}(t)$ is a straight line solution. Otherwise, $\bar{V}_1$ is an eigenvector.

Example: $\frac{d\bar{Y}}{dt} = A\bar{Y} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\bar{Y}_1$

A has repeated eigenvalue $\lambda = -2$

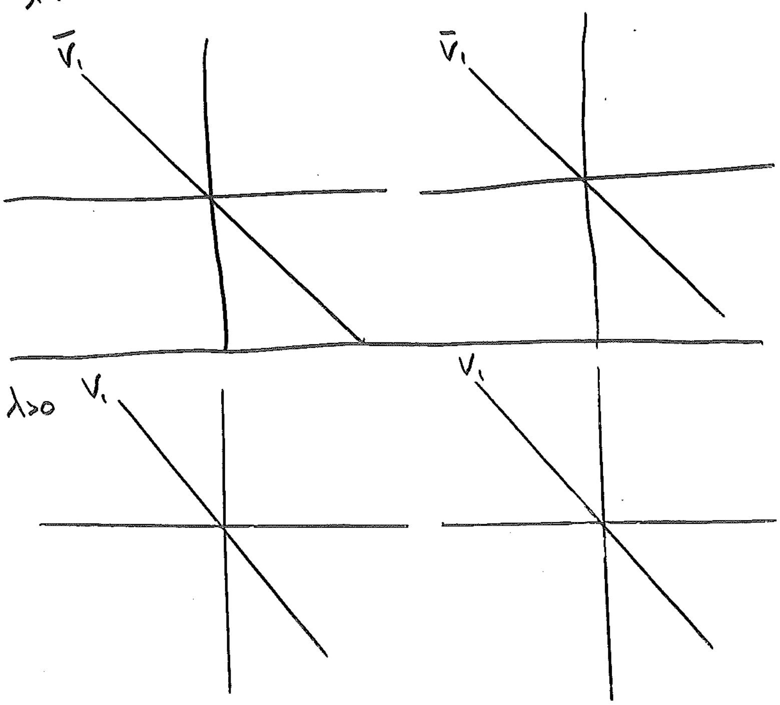
Let $\bar{V}_0 = \bar{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be an arbitrary initial condition.

$\bar{V}_1 =$

Qualitative Analysis of Systems with Repeated Eigenvalues

$$\bar{Y}(t) = e^{\lambda t}V_0 + te^{\lambda t}\bar{V}_1$$
$$= e^{\lambda t}(\bar{V}_0 + t\bar{V}_1)$$

$\lambda < 0$



Example A harmonic oscillator with repeated eigenvalues

$$\frac{d^2y}{dt^2} + 2\sqrt{2}\frac{dy}{dt} + 2y = 0$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 2\sqrt{2}\bar{v}$$

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{pmatrix}\bar{Y} \qquad \bar{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$$

$\det(B - \lambda I) = 0 \Rightarrow \lambda = -\sqrt{2}$ is repeated

eigenvector $\bar{V} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$

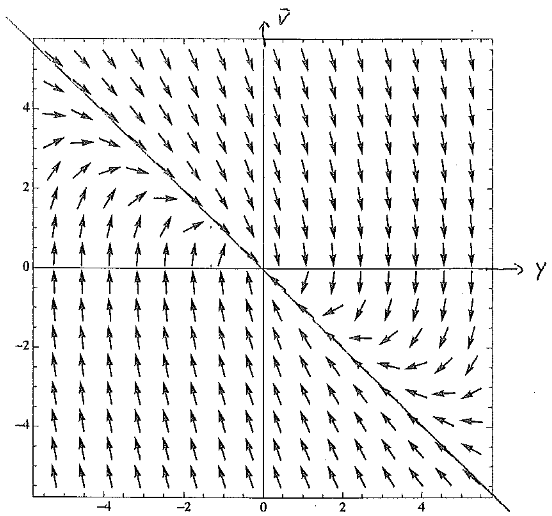
For arbitrary initial condition $\bar{V}_0 = (x_0, y_0)$

$$\bar{V}_1 = (B + \sqrt{2}I)\bar{V}_0 = \begin{pmatrix} \sqrt{2} & 1 \\ -2 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2}y_0 + v_0 \\ -2y_0 - \sqrt{2}v_0 \end{pmatrix}$$

$$\frac{d^2y}{dt^2} + 2\sqrt{2}\,\frac{dy}{dt} + 2y = 0$$

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{pmatrix}\bar{Y} \qquad \lambda = -\sqrt{2} \qquad \bar{V} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$



Systems with Zero as an Eigenvalue

Example: $$\frac{d\bar{Y}}{dt} = A\bar{Y} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} -3-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix} = 0$$

$$(-3-\lambda)(-1-\lambda) - 3 = 0$$

$$\lambda^2 + 3 + 4\lambda - 3 = 0$$

$$\lambda^2 + 4\lambda = 0$$

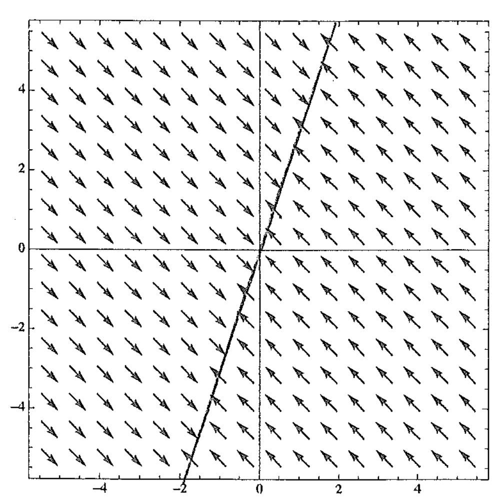
$$\lambda(\lambda + 4) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = -4$$

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}\bar{Y}$$

eigenvalues $\lambda_1 = 0$ $\qquad \lambda_2 = -4$

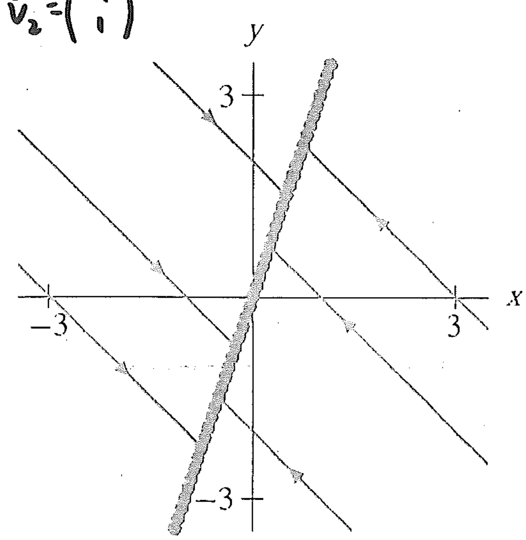
eigenvectors $\bar{V}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\qquad \bar{V}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}\bar{Y}$$

$\lambda_1 = 0$ $\qquad \lambda_2 = -4$

$\bar{V}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\qquad \bar{V}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



In general for $\frac{d\bar{Y}}{dt} = A\bar{Y}$

where A has eigenvalues $\lambda_1 = 0$, $\lambda_2 \neq 0$

with corresponding eigenvectors $\bar{V}_1$, $\bar{V}_2$

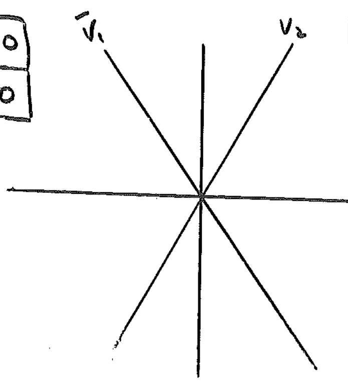
general solution

$$\bar{Y}(t) = k_1 e^{\lambda_1 t}\bar{V}_1 + k_2 e^{\lambda_2 t}\bar{V}_2$$

$$\bar{Y}(t) = k_1\bar{V}_1 + k_2 e^{\lambda_2 t}\bar{V}_2$$

Note: If $k_2 = 0$ $\bar{Y}(t) = k_1\bar{V}_1$ are equilibrium solutions

$\boxed{\lambda_2 < 0}$
$\boxed{\lambda_1 = 0}$



$\boxed{\lambda_2 > 0,\ \lambda_1 = 0}$