

Sec 3.5 Special Cases: Repeated + zero Eigenvalues

$$\frac{d\bar{y}}{dt} = A\bar{y} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \bar{y}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{pmatrix} = 0$$

$$(-2-\lambda)(-2-\lambda) = 0$$

$$(\lambda+2)^2 = 0$$

$$\boxed{\lambda = -2}$$

Find corresponding eigenvector

$$A\bar{v} = -2\bar{v}$$

$$\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -2 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} -2x + y = -2x & y=0 \\ -2y = -2y \end{cases}$$

eigenvector $\begin{pmatrix} x \\ 0 \end{pmatrix}$ or $x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Another approach

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \bar{y} \Leftrightarrow$$

$$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = -2y \end{cases} \left. \begin{array}{l} \text{partially} \\ \text{decoupled} \end{array} \right\}$$

$$\frac{dy}{dt} = y_0 e^{-2t}$$

$$(*) \quad \frac{dx}{dt} = -2x + y_0 e^{-2t}$$

$$\text{step 1} \quad x_h(t) = x_0 e^{-2t} \quad \text{solves} \quad \frac{dx}{dt} = -2x$$

$$\text{step 2} \quad x_p(t) = A t e^{-2t}$$

$$\text{plug into} \quad A e^{-2t} - 2A t e^{-2t} = -2A t e^{-2t} + y_0 e^{-2t}$$

$$\Rightarrow A = y_0$$

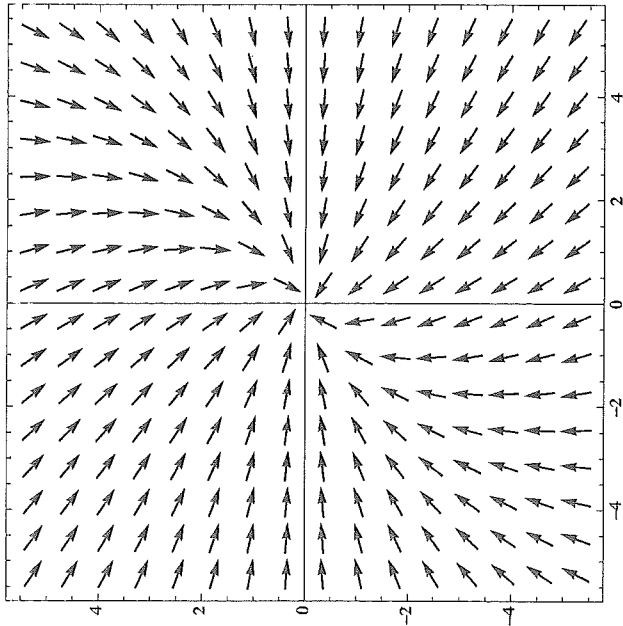
$$x(t) = x_h(t) + x_p(t) = x_0 e^{-2t} + y_0 t e^{-2t}$$

$$\bar{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{-2t} + y_0 t e^{-2t} \\ y_0 e^{-2t} \end{pmatrix}$$

$$\bar{y}(t) = e^{-2t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{-2t} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \bar{y} = A\bar{y}$$

repeated eigenvalue $\lambda = -2$
eigenvector $\bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



motivated by this example we try something of the same form.

If $\frac{d\bar{y}}{dt} = A\bar{y}$ is a system with repeated eigenvalue λ

Try a solution of the form

$$\bar{y}(t) = e^{\lambda t} \bar{V}_0 + t e^{\lambda t} \bar{V}_1,$$

where $\bar{V}_0 = \bar{y}(0)$, the initial condition for $\bar{y}(t)$

We plug into the equation $\frac{d\bar{y}}{dt} = A\bar{y}$

$$\frac{d\bar{y}}{dt} =$$

$$A\bar{y} =$$

equating the left and right hand side

Theorem: Suppose $\frac{d\bar{Y}}{dt} = A\bar{Y}$ is a linear system in which the 2×2 matrix A has a real repeated eigenvalue λ and only one line of eigenvectors.

Then the general solution has the form

$$\bar{Y}(t) = e^{\lambda t} \bar{V}_0 + t e^{\lambda t} \bar{V}_1$$

where $\bar{V}_0 = (x_0, y_0)$ is an arbitrary initial condition

and $\bar{V}_1 = (A - \lambda I) \bar{V}_0$.

If $\bar{V}_1 = \bar{0}$ then $\bar{V}_0$ is an eigenvector and $\bar{Y}(t)$ is a straight line solution. Otherwise, $\bar{V}_1$ is an eigenvector.

Example: $\frac{d\bar{Y}}{dt} = A\bar{Y} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \bar{Y}$

A has repeated eigenvalue $\lambda = -2$

Let $\bar{V}_0 = \bar{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be an arbitrary initial

condition.

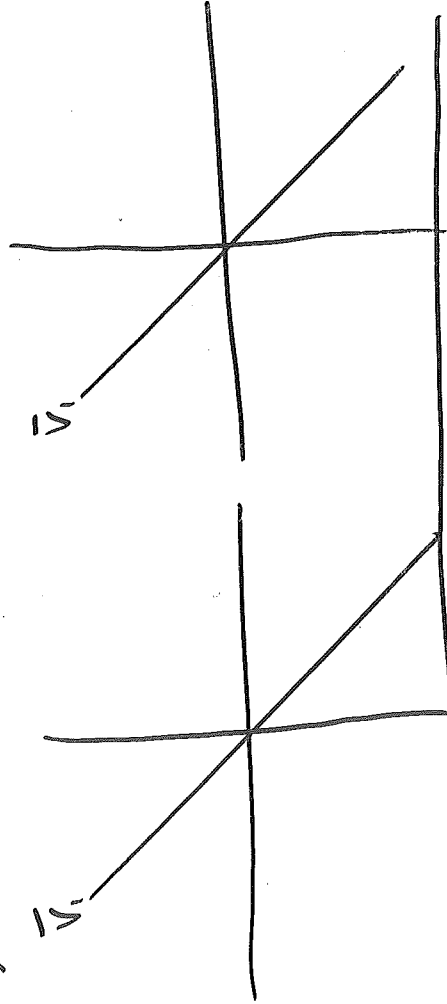
$$\bar{V}_1 =$$

Qualitative Analysis of Systems with Repeated

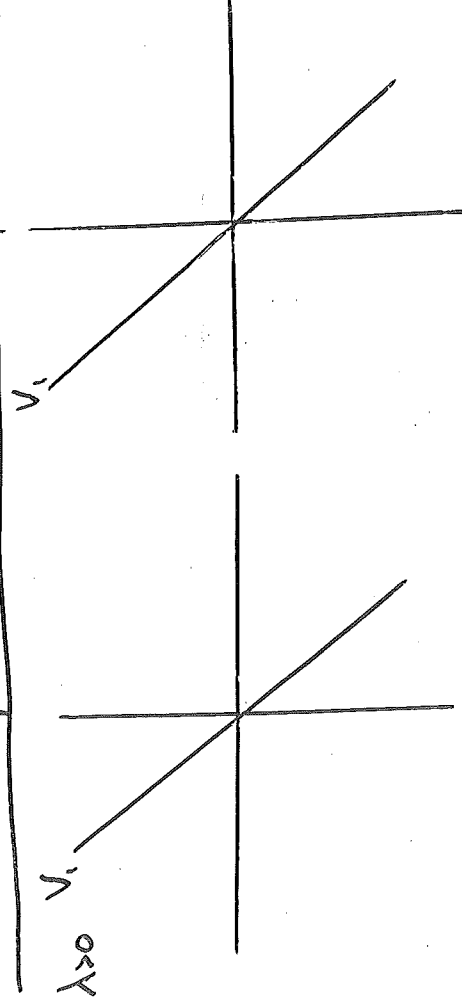
Eigenvalues

$$\begin{aligned}\bar{y}(t) &= e^{\lambda t} V_0 + t e^{\lambda t} \bar{V}_1 \\ &= e^{\lambda t} (\bar{V}_0 + t \bar{V}_1)\end{aligned}$$

$\lambda < 0$



$\lambda > 0$



Example A harmonic oscillator with repeated eigenvalues

$$\frac{d^2 y}{dt^2} + 2\sqrt{2} \frac{dy}{dt} + 2y = 0$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 2\sqrt{2}v$$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{pmatrix} \bar{y} \quad \bar{y} = \begin{pmatrix} y \\ v \end{pmatrix}$$

$$\det(B - \lambda I) = 0 \Rightarrow \lambda = -\sqrt{2} \text{ is repeated}$$

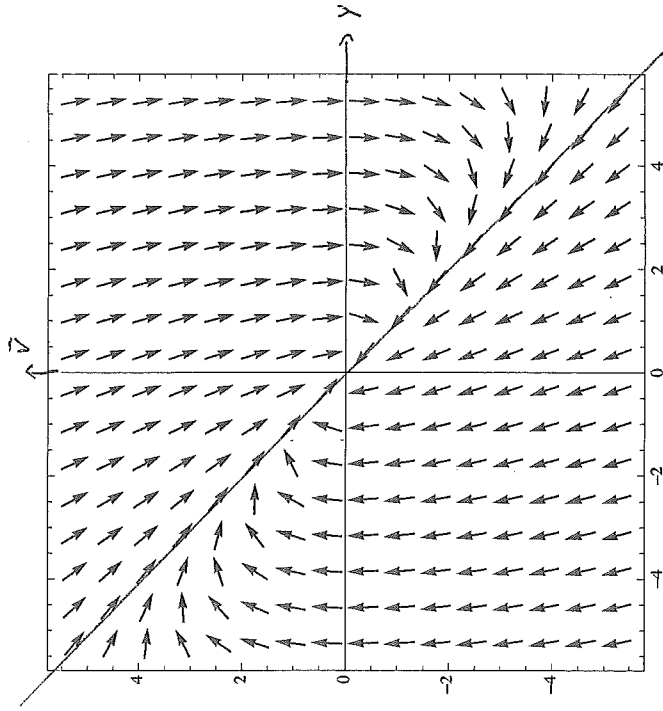
$$\text{eigenvector } \bar{V} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

For arbitrary initial condition $\bar{V}_0 = (x_0, y_0)$

$$\begin{aligned}\bar{V}_1 &= (B + \sqrt{2}I)\bar{V}_0 = \begin{pmatrix} \sqrt{2} & 1 \\ -2 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2}x_0 + y_0 \\ -2x_0 - \sqrt{2}y_0 \end{pmatrix}\end{aligned}$$

$$\frac{d^2 y}{dt^2} + 2\sqrt{2} \frac{dy}{dt} + 2y = 0$$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{pmatrix} \bar{y} \quad \lambda = -\sqrt{2} \quad \bar{V} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$



Systems with zero as an Eigenvalue

Example: $\frac{d\bar{y}}{dt} = A\bar{y} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \bar{y}$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} -3-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix} = 0$$

$$(-3-\lambda)(-1-\lambda) - 3 = 0$$

$$\lambda^2 + 3 + 4\lambda - 3 = 0$$

$$\lambda^2 + 4\lambda = 0$$

$$\lambda(\lambda + 4) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = -4$$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \bar{y}$$

eigenvalues

$$\lambda_1 = 0$$

$$\lambda_2 = -4$$

eigenvectors

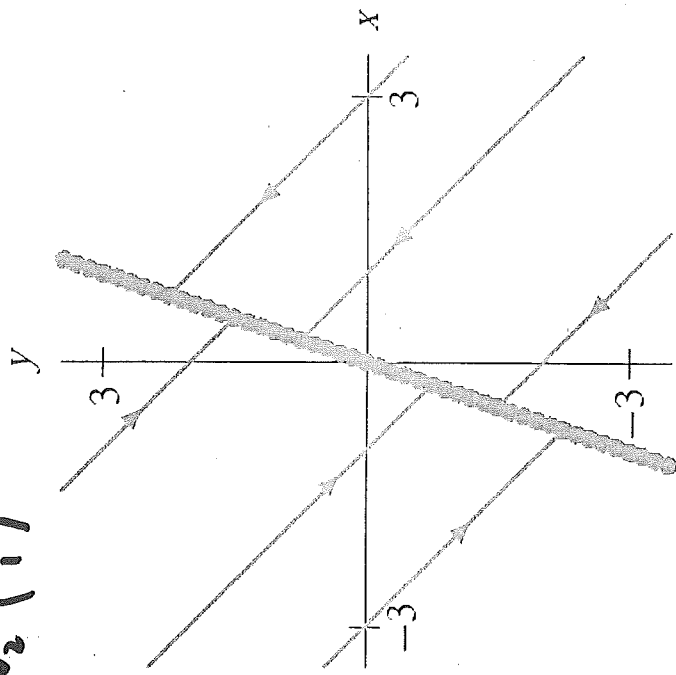
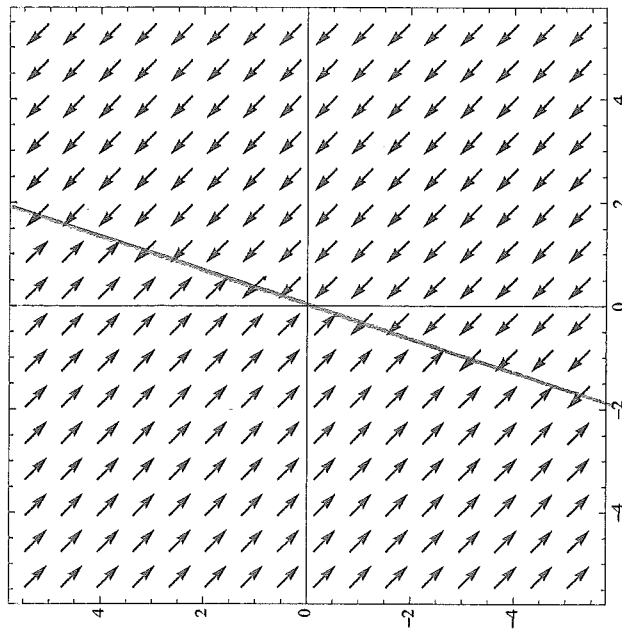
$$\bar{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\bar{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \bar{y}$$

$$\lambda_1 = 0 \quad \lambda_2 = -4$$

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



In general for $\frac{d\bar{y}}{dt} = A\bar{y}$

where A has eigenvalues $\lambda_1 = 0, \lambda_2 \neq 0$
with corresponding eigenvectors $\bar{v}_1, \bar{v}_2$

general solution

$$\bar{y}(t) = k_1 e^{\lambda_1 t} \bar{v}_1 + k_2 e^{\lambda_2 t} \bar{v}_2$$

$$\bar{y}(t) = k_1 \bar{v}_1 + k_2 e^{\lambda_2 t} \bar{v}_2$$

Note: If $k_2 = 0$ $\bar{y}(t) = k_1 \bar{v}_1$ are equilibrium

solutions

$\lambda_2 < 0$
$\lambda_1 = 0$

$\lambda_2 > 0, \lambda_1 = 0$

