Sec 3.2 Straight-Line Solutions

Recall the partially coupled system

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -4y$$

In vector form we have

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d\bar{Y}}{dt} = A\bar{Y} \quad \text{where } A = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}$$

$$\bar{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

In sec 2.4 P. we showed that

$$\left.\begin{aligned} x(t) &= k_1 e^{2t} - \tfrac{1}{2}k_2 e^{-4t} \\ y(t) &= k_2 e^{-4} \end{aligned}\right\} \text{is a general solution}$$

In particular we can choose $\begin{cases} k_1 = 0 \\ k_2 = 0 \end{cases}$ to see that

$$\begin{aligned} x(t) &= e^{2t} \\ y(t) &= 0 \end{aligned} \quad \text{is a solution}$$

and we can choose $k_1 = 0,\ k_2 = 1$ to see that

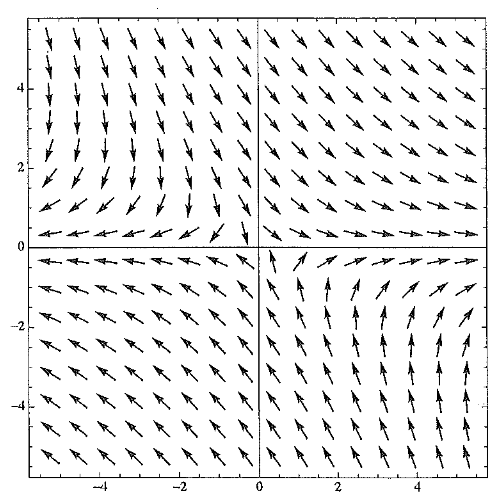
$$\begin{aligned} x(t) &= -\tfrac{1}{2}e^{-4t} \\ y(t) &= e^{-4t} \end{aligned} \quad \text{is also a solution}$$

Let $\bar{Y}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$ $\quad \bar{Y}_2 = \begin{pmatrix} -\frac{1}{2}e^{-4t} \\ e^{-4t} \end{pmatrix}$

Let's verify that these vector valued functions solve the linear system

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\bar{Y}$$

$$\frac{d\overline{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\vec{Y}$$



Note that $\bar{Y}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\bar{Y}_2(0) = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$ are linearly independent.

$\therefore$ Every solution for $\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\bar{Y}$

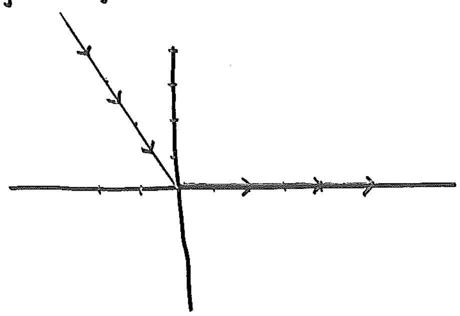
can be written as $k_1\bar{Y}_1(t) + k_2\bar{Y}_2(t)$

<u>Example</u>: what is the solution of

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\bar{Y} \quad \text{passing through } (-4, 2)$$

How to find <u>straight line solutions</u>

Observe that the vector field along the solution "curve" of a straight line solution points either directly towards $(0,0)$ or directly away from $(0,0)$



straight line solutions are made up of vectors $\bar{V} = \begin{pmatrix} x \\ y \end{pmatrix}$ where the vector field at $\begin{pmatrix} x \\ y \end{pmatrix}$ points either in the same direction as the vector from $(0,0)$ to (x,y) or in the opposite direction.

Mathematically this means

$$A\bar{V} = A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

If $\lambda > 0$ then the vector field points in the same direction as $(0,0)$ to (x,y), if $\lambda < 0$ then the vector field points in the opposite direction.

<u>Example</u>: $$\begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>Definition</u>: Given a matrix A, a number λ is called an <u>eigenvalue</u> of A if there is a nonzero vector $\bar{V} = (x,y)$ for which

$$A\bar{V} = A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \bar{V}$$

The vector $\bar{V}$ is called an <u>eigenvector</u>.

<u>Example</u>: $A = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix}$

$\bar{V} = (6,-2)$ is an eigenvector

<u>Lines of Eigenvectors</u>

Given a matrix A, if $\bar{V}$ is an eigenvector for eigenvalue λ, then any scalar multiple $k\bar{V}$ is also an eigen vector

$$A(k\bar{V}) =$$

$\therefore$ Given an eigenvector $\bar{V}$ for the eigenvalue λ the entire line of vectors through $\bar{V}$ and the origin are also eigenvectors for λ.

<u>Computation of Eigenvalues</u>

Ask for $\bar{V}$ and λ so that

$$A\bar{V} = \lambda \bar{V}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} ax + by &= \lambda x \\ cx + dy &= \lambda y \end{aligned}$$

$$(a-\lambda)x + by = 0$$
$$cx + (d-\lambda)y = 0 \qquad (*)$$

If $\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$

There will be non-trivial solutions of $(*)$

If $(a-\lambda)(d-\lambda) - b\cdot c = 0$

$$a\cdot d - a\lambda - d\lambda + \lambda^2 - bc = 0$$
$$\underbrace{\lambda^2 - (a+d)\lambda + (ad-bc)}_{\text{characteristic polynomial}} = 0$$

The Matrix Approach to the Characteristic Polynomial

Computation of Eigenvectors

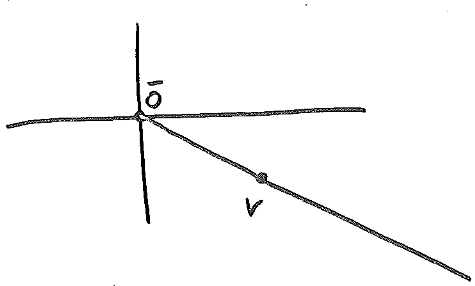
Example: $B = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

Step 1: Find eigenvalues

Fact: Let A be a 2×2 matrix with eigenvalue λ and corresponding eigenvector $\bar{V}$.

① $\bar{Y}(t) = e^{\lambda t}\bar{V} \qquad -\infty < t < \infty$

represents all the points on a ray that originates at the origin and passes through the point V.



② $\bar{Y}(t) = e^{\lambda t}\bar{V}$ is a solution of

$$\frac{d\bar{Y}}{dt} = A\bar{Y}$$

Verify

$$\frac{d}{dt}(\bar{Y}) = \frac{d}{dt}\left(e^{\lambda t}\bar{V}\right) = \lambda e^{\lambda t} V$$

Fact: Suppose the matrix A has two distinct real eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\bar{V}_1$ and $\bar{V}_2$. Then the vector valued functions

$$\bar{Y}_1(t) = e^{\lambda_1 t}\bar{V}_1 \quad \text{and} \quad \bar{Y}_2(t) = e^{\lambda_2 t} V_2$$

are both solutions of $\frac{d\bar{Y}}{dt} = A\bar{Y}$

Furthermore, $\bar{Y}_1(0) = \bar{V}_1$ and $\bar{Y}_2(0) = \bar{V}_2$ are linearly independent.

Therefore the general solution of $\frac{d\bar{Y}}{dt} = A\bar{Y}$ is

$$\bar{Y}(t) = k_1\bar{Y}_1(t) + k_2\bar{Y}_2(t)$$
$$= k_1 e^{\lambda_1 t}\bar{V}_1 + k_2 e^{\lambda_2 t}\bar{V}_2$$

Example: Solve the initial value problem

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\bar{Y} \qquad \bar{Y}(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\vec{Y}$$