

Sec 3.2 Straight-Line Solutions

Recall the partially coupled system

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -4y$$

In vector form we have

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d\bar{y}}{dt} = A\bar{y} \quad \text{where } A = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}$$

$$\bar{y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

In sec 2.4 p. we showed

that $\left. \begin{aligned} x(t) &= k_1 e^{2t} - \frac{1}{2} k_2 e^{-4t} \\ y(t) &= k_2 e^{-4t} \end{aligned} \right\} \text{ is a general solution}$

In particular we can choose $\begin{cases} k_1 = 0 \\ k_2 = 0 \end{cases}$ to see that $\begin{aligned} x(t) &= e^{2t} \\ y(t) &= 0 \end{aligned}$ is a solution

and we can choose $k_1 = 0, k_2 = 1$ to see that

$$x(t) = -\frac{1}{2} e^{-4t}$$

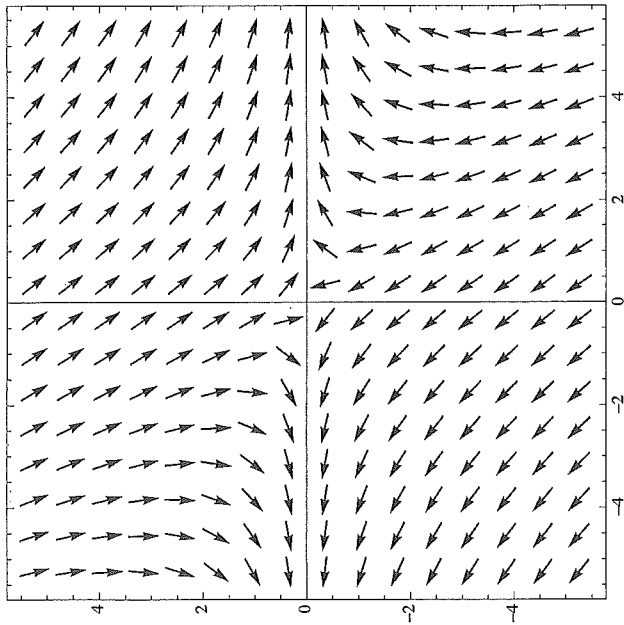
$$y(t) = e^{-4t} \quad \text{is also a solution}$$

$$\text{Let } \bar{y}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \quad \bar{y}_2 = \begin{pmatrix} -\frac{1}{2} e^{-4t} \\ e^{-4t} \end{pmatrix}$$

Let's verify that these vector valued functions solve the linear system

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \bar{y}$$

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \bar{Y}$$



Note that $\bar{Y}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\bar{Y}_2(0) = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$ are linearly independent.

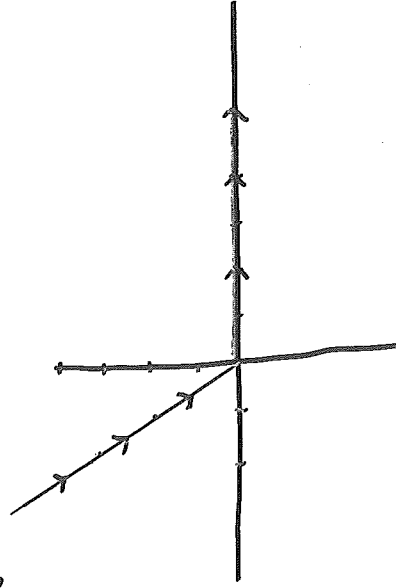
$\therefore$ Every solution for $\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \bar{Y}$ can be written as $k_1 \bar{Y}_1(t) + k_2 \bar{Y}_2(t)$

Example: what is the solution of

$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \bar{Y}$ passing through $(-4, 2)$

How to find straight line solutions

Observe that the vector field along the solution "curve" of a straight line solution points either directly towards $(0,0)$ or directly away from $(0,0)$



straight line solutions are made up of vectors $\vec{V} = \begin{pmatrix} x \\ y \end{pmatrix}$ where the vector field at $\begin{pmatrix} x \\ y \end{pmatrix}$ points either in the same direction as the vector from $(0,0)$ to (x,y) or in the opposite direction.

Mathematically this means

$$A\vec{V} = A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

If $\lambda > 0$ then the vector field points in the same direction as $(0,0)$ to (x,y) , if $\lambda < 0$ then the vector field points in the opposite direction.

Example: $\begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

Definition: Given a matrix A , a number λ is called an eigenvalue of A if there is a nonzero vector $\bar{V} = (x, y)$ for which

$$A\bar{V} = A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \bar{V}$$

The vector $\bar{V}$ is called an eigenvector.

Example: $A = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix}$

$\bar{V} = (6, -2)$ is an eigenvector

Lines of Eigenvectors

Given a matrix A , if $\bar{V}$ is an eigenvector for eigenvalue λ , then any scalar multiple $k\bar{V}$ is also an eigen vector

$$A(k\bar{V}) =$$

$\therefore$ Given an eigenvector $\bar{V}$ for the eigenvalue λ the entire line of vectors through $\bar{V}$ and the origin are also eigenvectors for λ .

Computation of Eigenvalues

Ask for $\bar{V}$ and λ so that

$$A\bar{V} = \lambda \bar{V}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$ax + by = \lambda x$$

$$cx + dy = \lambda y$$

Computation of Eigenvectors

Example: $B = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

Step 1: Find eigenvalues

$$(a-\lambda)x + by = 0 \quad (*)$$

$$cx + (d-\lambda)y = 0$$

$$\text{If } \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

There will be non-trivial solutions of (*)

$$\text{If } (a-\lambda)(d-\lambda) - b \cdot c = 0$$

$$a \cdot d - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

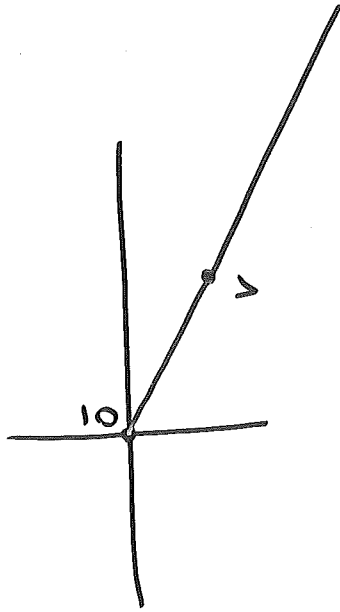
characteristic polynomial

The Matrix Approach to the Characteristic Polynomial

Fact: Let A be a 2×2 matrix with eigenvalue λ and corresponding eigenvector $\bar{V}$.

$$\textcircled{1} \quad \bar{Y}(t) = e^{\lambda t} \bar{V} \quad -\infty < t < \infty$$

represents all the points on a ray that originates at the origin and passes through the point V .



$$\textcircled{2} \quad \bar{Y}(t) = e^{\lambda t} \bar{V} \quad \text{is a solution of}$$

$$\frac{d\bar{Y}}{dt} = A\bar{Y}$$

Verify

$$\frac{d}{dt}(\bar{Y}) = \frac{d}{dt}(e^{\lambda t} \bar{V}) = \lambda e^{\lambda t} \bar{V}$$

Fact: Suppose the matrix A has two distinct real eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\bar{V}_1$ and $\bar{V}_2$. Then the vector valued functions

$$\bar{Y}_1(t) = e^{\lambda_1 t} \bar{V}_1 \quad \text{and} \quad \bar{Y}_2(t) = e^{\lambda_2 t} \bar{V}_2$$

are both solutions of $\frac{d\bar{Y}}{dt} = A\bar{Y}$

Furthermore, $\bar{Y}_1(0) = \bar{V}_1$ and $\bar{Y}_2(0) = \bar{V}_2$

are linearly independent.

Therefore the general solution of $\frac{d\bar{Y}}{dt} = A\bar{Y}$

$$\begin{aligned} \text{is } \bar{Y}(t) &= k_1 \bar{Y}_1(t) + k_2 \bar{Y}_2(t) \\ &= k_1 e^{\lambda_1 t} \bar{V}_1 + k_2 e^{\lambda_2 t} \bar{V}_2 \end{aligned}$$

Example: Solve the initial value problem

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \bar{Y} \quad \bar{Y}(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \bar{Y}$$

