Sec 3.1 Properties of Linear Systems

In this chapter we are going to study in great detail the behavior of linear systems with constant coefficients. Specifically systems of the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

We will use matrix and vector notation to express the system as

$$\frac{d\bar{Y}}{dt} = A\bar{Y}$$

where $\bar{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Equilibrium Points

What values of $\bar{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$ make the right hand side of

$$\frac{d\bar{Y}}{dt} = A\bar{Y}$$

equal to zero?

$$A\bar{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$ax + by = 0$$
$$cx + dy = 0$$

Fact: If $\det A = 0$ then $A\bar{Y} = \bar{0}$ has the unique solution $\bar{Y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

The Linearity Principle

Suppose $\frac{d\bar{Y}}{dt} = A\bar{Y}$ is a linear system of differential equations

1.) If $\bar{Y}_1(t)$ is a solution of this system, then $k\bar{Y}_1(t)$ is also a solution

2.) If $\bar{Y}_1(t)$ and $\bar{Y}_2(t)$ are two solutions of this system, then $\bar{Y}_1(t) + \bar{Y}_2(t)$ is also a solution.

Recall the partially decoupled system from Sec 2.4

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -4y$$

Matrix Facts

1.) If A is a 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and $\bar{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$A(k\bar{Y}) = kA\bar{Y}$ for any scalar k

Matrix Facts

2.) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\bar{Y}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ $\bar{Y}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Then $A(\bar{Y}_1 + \bar{Y}_2) = A\bar{Y}_1 + A\bar{Y}_2$

Verifying the Linearity Principle

If $\bar{Y}_1(t)$ and $\bar{Y}_2(t)$ are solutions of

$$\frac{d\bar{Y}}{dt} = A\bar{Y}$$

1.) $\dfrac{d(k\bar{Y}_1)}{dt} =$

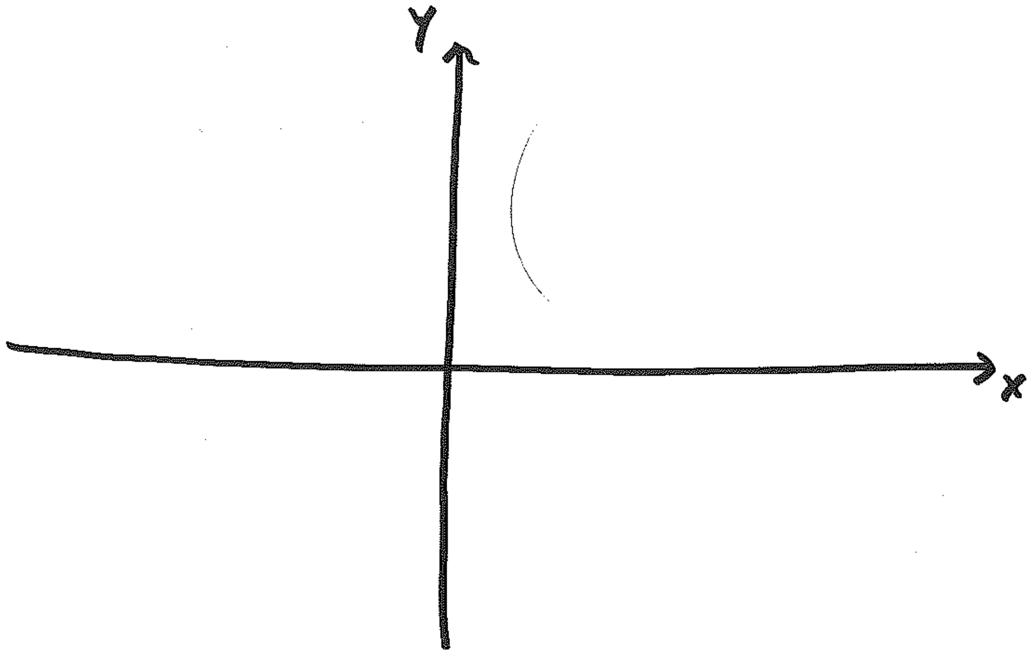
2.) $\dfrac{d(\bar{Y}_1 + \bar{Y}_2)}{dt} =$

<u>Definition</u>: Two vectors are <u>linearly independent</u> if they are not scalar multiples of each other.

If there exists a scalar k so that

$$\bar{Y}_2 = k\,\bar{Y}_1 \quad \text{then} \quad \bar{Y}_1 \text{ and } \bar{Y}_2$$

are <u>NOT</u> linearly independent.



Theorem: Suppose that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are two linearly independent vectors in the plane. Then given **any** vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ there exists scalars k_1 and k_2 so that

$$k_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Example: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ are linearly independent. Find k_1 and k_2 so that

$$k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Theorem for the General Solution of a linear system:

<u>Theorem</u>: Suppose $\bar{Y}_1(t)$ and $\bar{Y}_2(t)$ are solutions of the linear system $\frac{d\bar{Y}}{dt} = A\bar{Y}$. If $\bar{Y}_1(0)$ and $\bar{Y}_2(0)$ are linearly independent, then for any initial condition $\bar{Y}(0) = (x_0, y_0)$ we can find constants k_1 and k_2 so that $k_1\bar{Y}_1(t) + k_2\bar{Y}_2(t)$ is a solution of the initial value problem

$$\frac{d\bar{Y}}{dt} = A\bar{Y}, \quad \bar{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Example: Solve the initial value problem

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \bar{Y} \qquad \bar{Y}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$