

Sec 2.2 - The Geometry of Systems

Goal: To rewrite a system of differential equations in vector form.

Why? simplify both the notation and the idea of visualizing the changing of a single quantity

Example: $\frac{dR}{dt} = 2R - 1.2RF$

$$\frac{dF}{dt} = -F + 0.9RF$$

Let $\vec{P}(t) = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}$ then $\frac{d\vec{P}}{dt} = \begin{pmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{pmatrix}$

$$\frac{d\vec{P}}{dt} = \begin{pmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{pmatrix} = \begin{pmatrix} 2R - 1.2RF \\ -F + 0.9RF \end{pmatrix} = \vec{V} \begin{pmatrix} R \\ F \end{pmatrix} = \vec{V}(\vec{P})$$

$$\frac{d\vec{P}}{dt} = \vec{V}(\vec{P})$$

Vector Field for Simple Harmonic Oscillator

$$\frac{d^2 y}{dt^2} + \frac{k}{m} y = 0$$

Let $\frac{k}{m} = 1$ we have the corresponding system

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -y$$

Let $\vec{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$ Let $\vec{F}(\vec{Y}) = \vec{F} \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -y \end{pmatrix}$

Written in vector form we have

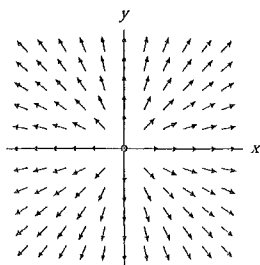
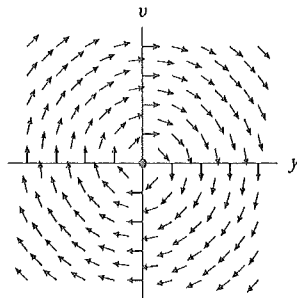
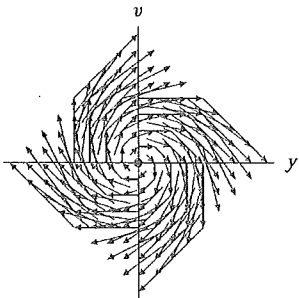
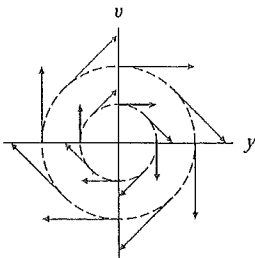
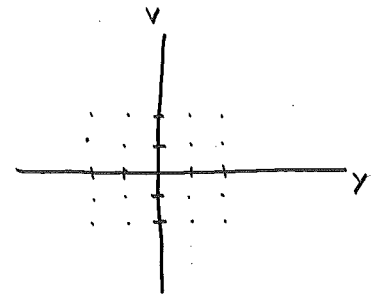
$$\frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y})$$

Note: $\vec{F} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\vec{F} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{F} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\vec{F} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$



In general, for a system with two dependent variables of the form

$$\frac{dx}{dt} = f(x, y) \quad (*)$$

$$\frac{dy}{dt} = g(x, y)$$

We introduce the vector $\vec{Y}(t) = (x(t), y(t))$ and the vector field $\vec{F}(\vec{Y}) = \vec{F}(x, y) = (f(x, y), g(x, y))$

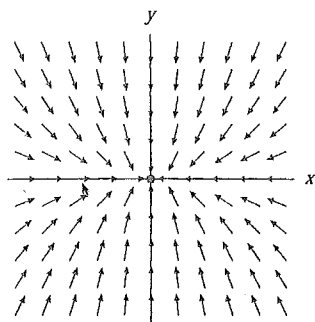
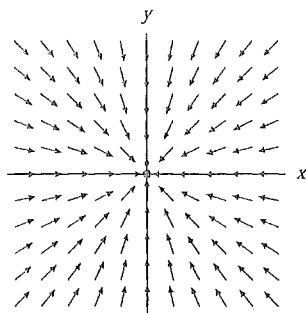
with this notation the system (*) of two equations may be written as

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \vec{F}(\vec{Y})$$

Examples: $\vec{F}(x, y) = (x, y)$

$$\vec{F}(x, y) = (-x, -y)$$

$$\vec{F}(x, y) = (-x, -2y)$$



Equilibrium Solutions

Definition: The point $\bar{y}_0$ is an equilibrium point for the system $\frac{d\bar{y}}{dt} = \bar{F}(\bar{y})$ if $\bar{F}(\bar{y}_0) = \bar{0}$.

The constant function $\bar{y}(t) = \bar{y}_0$ is an equilibrium point.

Example: A Population Model for Two Competing Species

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$

$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy$$

Equilibrium Points

$$\begin{cases} 2x\left(1 - \frac{x}{2}\right) - xy = 0 \\ 3y\left(1 - \frac{y}{3}\right) - 2xy = 0 \end{cases} \Rightarrow \begin{cases} x(2 - x - y) = 0 \\ y(3 - y - 2x) = 0 \end{cases}$$

Four Equilibrium points: $(0,0), (0,3), (2,0), (1,1)$

