Sec 2.2 - The Geometry of Systems

Goal: To rewrite a system of differential equations in vector form.

Why? Simplify both the notation and the idea of visualizing the changing of a single quantity

Example:

$$\frac{dR}{dt} = 2R - 1.2RF$$

$$\frac{dF}{dt} = -F + 0.9RF$$

Let $\vec{P}(t) = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}$ then $\dfrac{d\vec{P}}{dt} = \begin{pmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{pmatrix}$

$$\frac{d\vec{P}}{dt} = \begin{pmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{pmatrix} = \begin{pmatrix} 2R - 1.2RF \\ -F + 0.9RF \end{pmatrix} = \vec{V}\begin{pmatrix} R \\ F \end{pmatrix} = \vec{V}(P)$$

$$\frac{d\vec{P}}{dt} = \vec{V}(\vec{P})$$

Vector Field for Simple Harmonic Oscillator

$$\frac{d^2y}{dt^2} + \frac{k}{m} y = 0$$

Let $\frac{k}{m} = 1$ we have the corresponding system

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -y$$

Let $\bar{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$ Let $\bar{F}(\bar{Y}) = \bar{F}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -y \end{pmatrix}$

Written in vector form we have

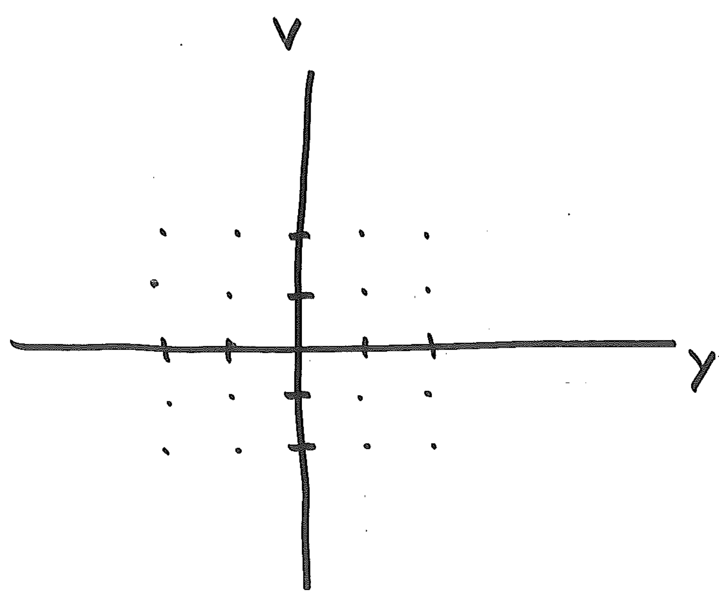
$$\frac{d\bar{Y}}{dt} = \vec{F}(\bar{Y})$$

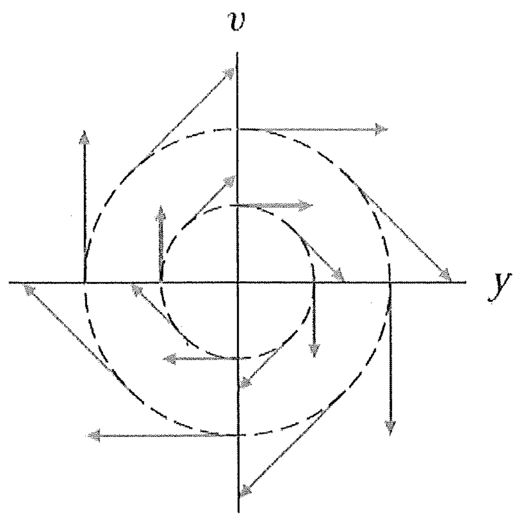
Note: $\bar{F}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

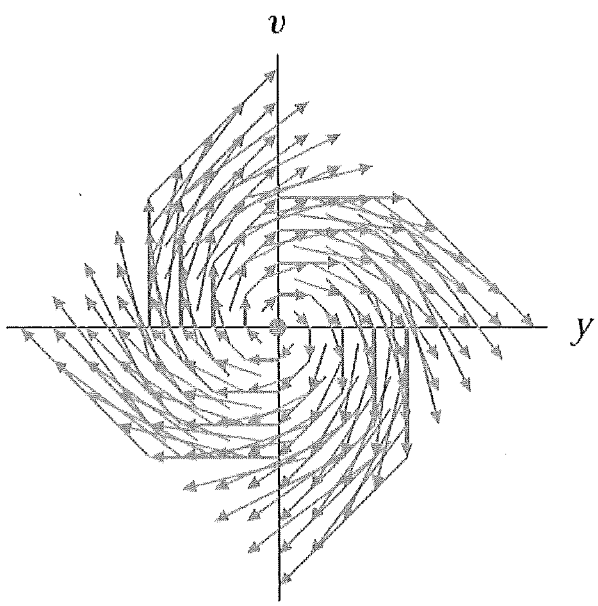
$\bar{F}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

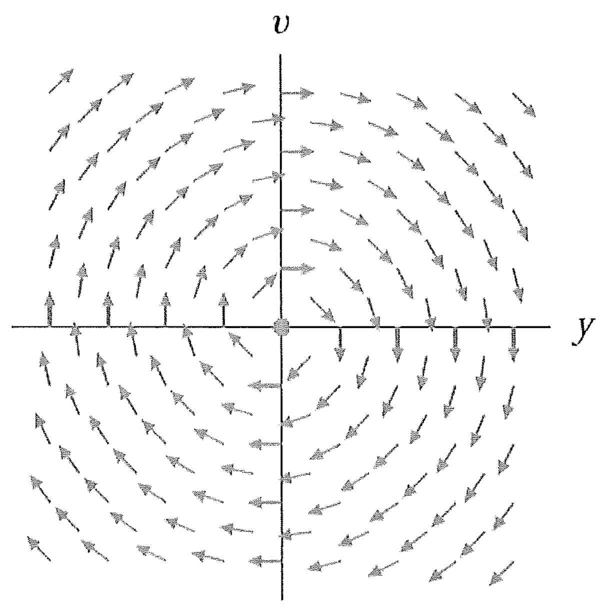
$\bar{F}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

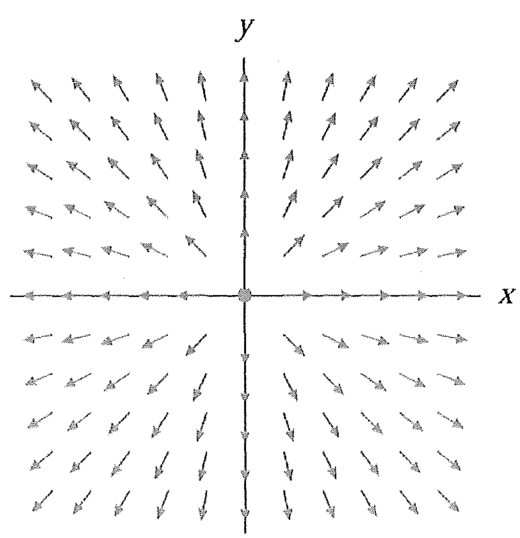
$F\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$




v
y


v
y


v
y


y
x

In general, for a system with two dependent variables of the form

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y) \qquad (*)$$

We introduce the vector $\bar{y}(t) = (x(t), y(t))$ and the vector field $\bar{F}(\bar{y}) = \bar{F}(x,y) = (f(x,y), g(x,y))$

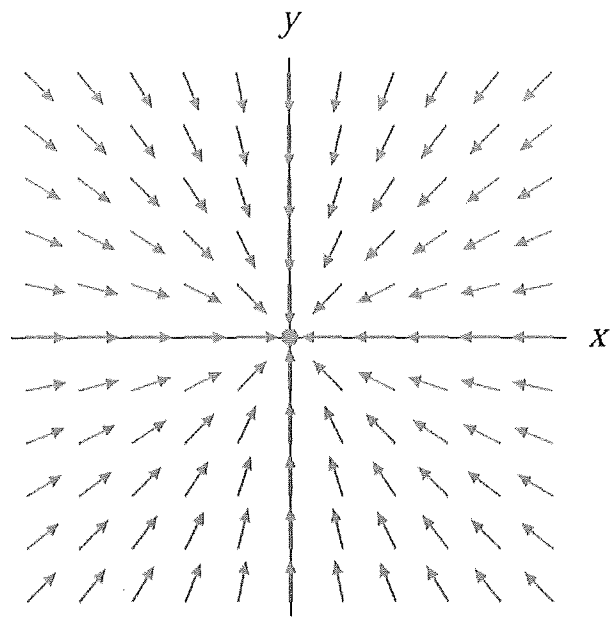
with this notation the system $(*)$ of two equations may be written as

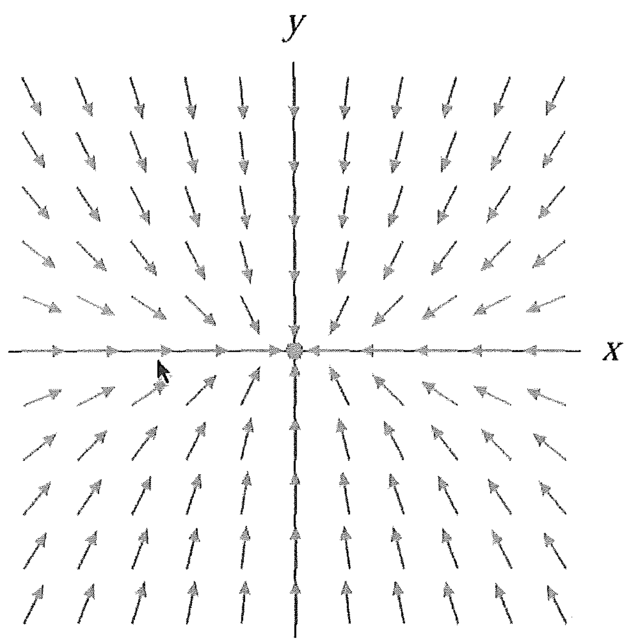
$$\frac{d\bar{y}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \vec{F}(\vec{y})$$

<u>Examples</u>: $\vec{F}(x,y) = (x,y)$

$\vec{F}(x,y) = (-x,-y)$

$\vec{F}(x,y) = (-x,-2y)$


y
x


y
x

Equilibrium Solutions

Definition: The point $\bar{Y}_0$ is an equilibrium point for the system $\frac{d\bar{Y}}{dt} = \bar{F}(\bar{Y})$ if $\bar{F}(\bar{Y}_0) = \bar{0}$.

The constant function $\bar{Y}(t) = \bar{Y}_0$ is an equilibrium point.

Example: A Population Model for Two Competing Species

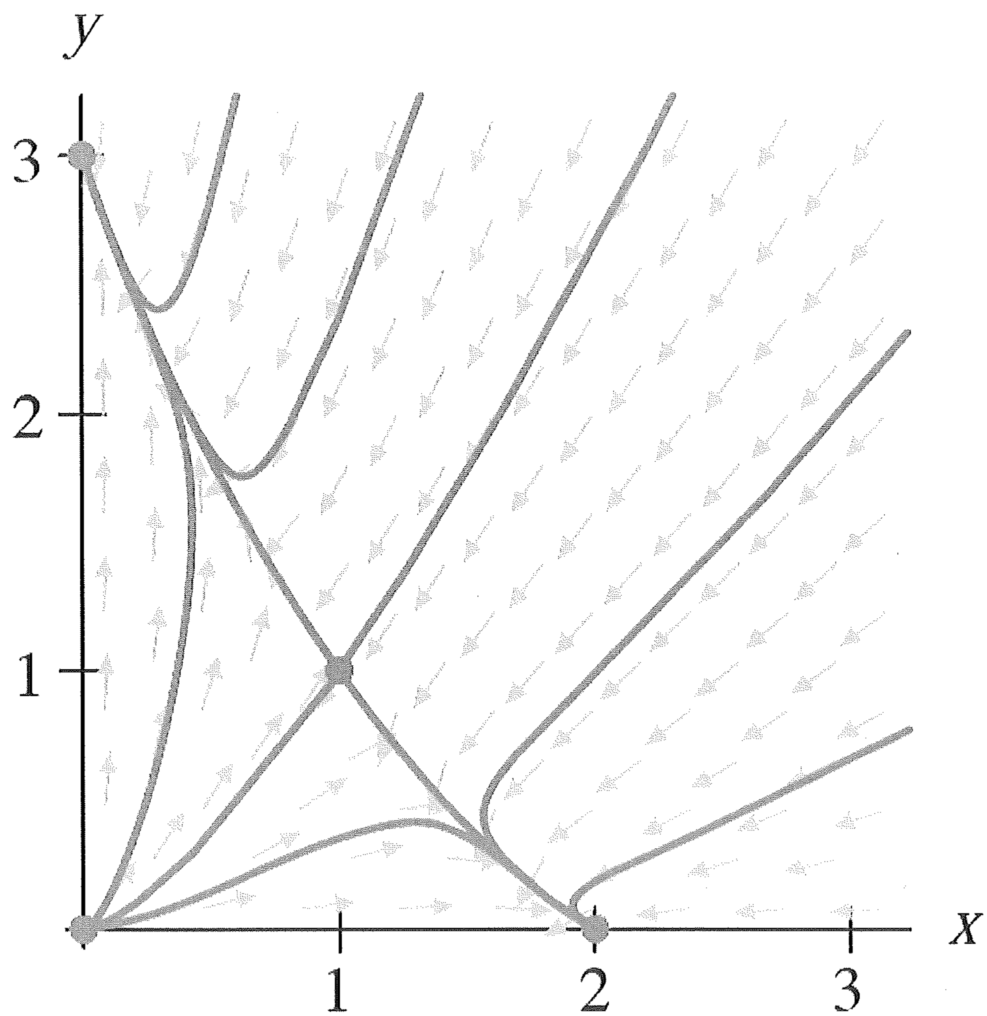
$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$

$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy$$

Equilibrium Points

$$\left.\begin{aligned} 2x\left(1-\frac{x}{2}\right) - xy &= 0 \\ 3y\left(1-\frac{y}{3}\right) - 2xy &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x(2 - x - y) &= 0 \\ y(3 - y - 2x) &= 0 \end{aligned}$$

Four Equilibrium points: $(0,0), (0,3), (2,0), (1,1)$



y
3
2
1
1
2
3
X