

Sec 2.3 Damped Harmonic Oscillator

The simple harmonic oscillator was derived from Hooke's Law

$$m \frac{d^2 y}{dt^2} = -ky$$

$$m, k > 0$$

The solutions are periodic. There is no damping.

A damping force is proportional to the velocity of the object and always acts in a direction which is opposite to the direction of motion.

$$\left. \begin{aligned} m \frac{d^2 y}{dt^2} &= -ky - b \frac{dy}{dt} \\ m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky &= 0 \end{aligned} \right\} \begin{array}{l} \text{Damped} \\ \text{Harmonic} \\ \text{Oscillator} \end{array}$$

To simplify the equation we divide by m

Recall $m > 0$.

$$\frac{d^2 y}{dt^2} + \frac{b}{m} \frac{dy}{dt} + \frac{k}{m} y = 0$$

$$\text{or } \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0$$

where $p, q > 0$

Guessing Solutions

Example:

$$\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 0$$

Classifying Solutions for the Damped Harmonic Oscillator

$$(*) \quad m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0 \quad m, b, k > 0$$

$$\text{Try } y(t) = e^{st} \quad y'(t) = se^{st} \quad y''(t) = s^2 e^{st}$$

Plug into (*)

$$ms^2 e^{st} + b \cdot se^{st} + ke^{st} = 0$$

$$e^{st} (ms^2 + b \cdot s + k) = 0$$

characteristic polynomial $ms^2 + bs + k$

$$\text{with roots} \quad s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

- When $b^2 - 4mk > 0$ two distinct real roots (always negative). Oscillator is called over damped.

- When $b^2 - 4mk < 0$, two complex solution with negative real part $-\frac{b}{2m}$. The oscillator is called underdamped.

- When $b^2 - 4mk = 0$, one repeated solution and the oscillator is called critically damped.

