Sec 2.3 Damped Harmonic Oscillator

The simple harmonic oscillator was derived from Hooke's Law

$$m\frac{d^2y}{dt^2} = -ky \qquad m, k > 0$$

The solutions are periodic. There is no damping.

A damping force is proportional to the velocity of the object and always acts in a direction which is opposite to the direction of motion.

$$\left.\begin{aligned} m\frac{d^2y}{dt^2} &= -ky - b\frac{dy}{dt} \\ \\ m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky &= 0 \end{aligned}\right\} \begin{array}{l}\text{Damped} \\ \text{Harmonic} \\ \text{Oscillator}\end{array}$$

To simplify the equation we divide by m

Recall $m > 0$.

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$$

or $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

where $p, q > 0$

Guessing Solutions

Example:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

Classifying Solutions for the Damped Harmonic Oscillator

$$(*) \qquad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \qquad m, b, k > 0$$

Try $y(t) = e^{st}$ $\quad y'(t) = se^{st}$ $\quad y''(t) = s^2e^{st}$

Plug into $(*)$

$$ms^2e^{st} + b \cdot se^{st} + ke^{st} = 0$$

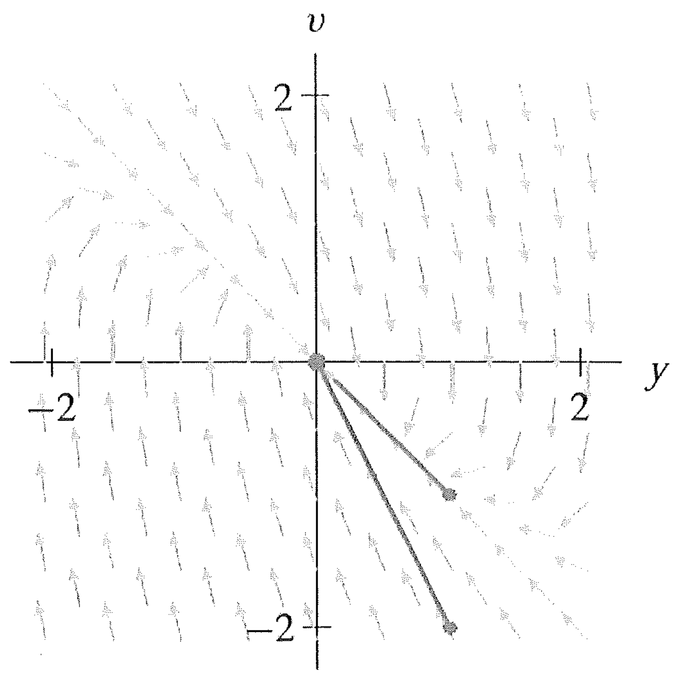
$$e^{st}\left(ms^2 + b \cdot s + k\right) = 0$$

characteristic polynomial $ms^2 + bs + k$

with roots $\qquad s = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

- when $b^2 - 4mk > 0$ two distinct real roots (always negative). Oscillator is called <u>over damped</u>.

- when $b^2 - 4mk < 0$, two complex solution with negative real part $-\frac{b}{2m}$. The oscillator is called <u>underdamped</u>.
- When $b^2 - 4mk = 0$, one repeated solution and the oscillator is called <u>critically damped</u>.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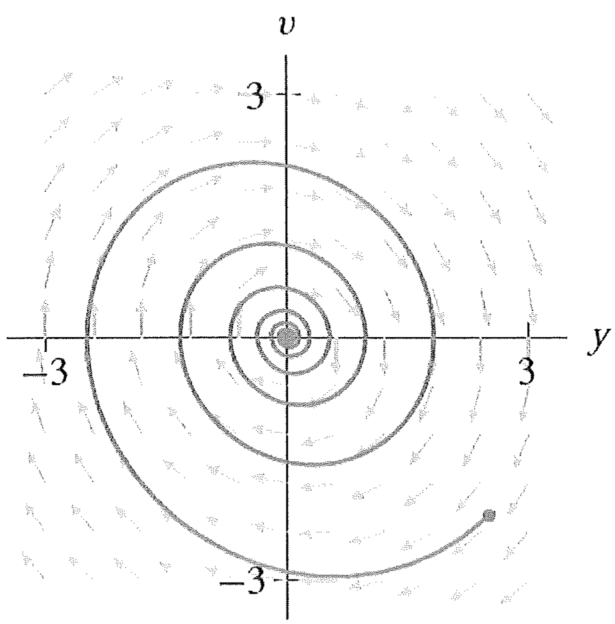
2
−2
y


v
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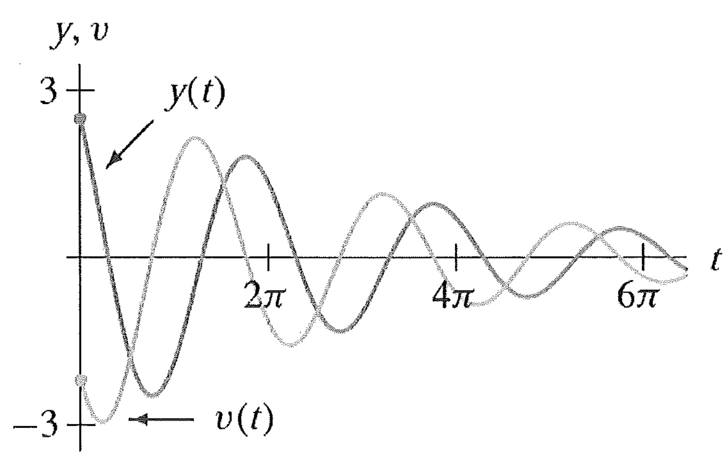
y, v
3
y(t)
2π
4π
6π
t
v(t)
−3