

sec 3.4 Systems without straight-Line Solutions

Example 1 $\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{y} = A\bar{y}$

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

Example 2 $\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \bar{y} = B\bar{y}$

$$\det(B - \lambda I) = \det \begin{pmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{pmatrix} = 0$$

$$(-2-\lambda)(-2-\lambda) + 9 = 0$$

$B = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$ has eigenvalues
 $\lambda_1 = -2+3i$ $\lambda_2 = -2-3i$

Let's Find the corresponding eigenvectors

For $\lambda_1 = -2+3i$, we want vector $\bar{V}_1$ such that

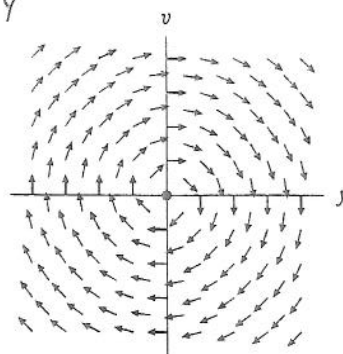
$$B\bar{V}_1 = \lambda_1 \bar{V}_1 \quad \text{let } \bar{V}_1 = (x_1, y_1)$$

$$\begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = (-2+3i) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Example 1

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{y}$$

$$\begin{cases} \frac{dy}{dt} = y \\ \frac{dx}{dt} = -y \end{cases}$$



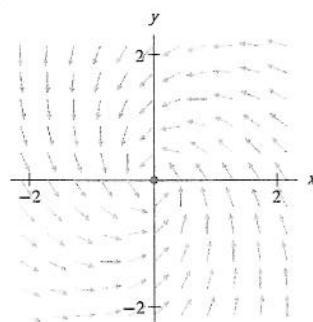
A solution

$$y(t) = \cos t$$

$$x(t) = -\sin t$$

Example 2

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \bar{y}$$



A solution

$$x(t) = -e^{-2t} \sin(3t)$$

$$y(t) = e^{-2t} \cos(3t)$$

Note: If λ is an eigenvalue of B and $\bar{V}$ is the corresponding eigenvector so that $B\bar{V} = \lambda\bar{V}$ (where λ and $\bar{V}$ are complex)

then $\bar{y}(t) = e^{\lambda t} \bar{V}$ is a solution of

$$\frac{d\bar{y}}{dt} = B\bar{y}$$

We have shown that

$$\bar{y}(t) = e^{(-2+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad (*)$$

is a solution of $\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \bar{y}$

Problem: We don't want complex valued solutions.

Euler's Formula: $e^{i\theta} = \cos \theta + i \sin \theta$

We can rewrite $\bar{y}(t)$ using Euler's Formula

$$\bar{y}(t) = e^{(-2+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{-2t} \cdot e^{3ti} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\bar{y}(t) = e^{(-2+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\bar{y}(t) = e^{-2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + i e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$$

$$\text{Let } \bar{y}_1(t) = \text{Re}(\bar{y}(t)) = e^{-2t} \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix}$$

$$\bar{y}_2(t) = \text{Im}(\bar{y}(t)) = e^{-2t} \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}$$

Note: $\bar{y}_1(t)$ and $\bar{y}_2(t)$ are real valued functions

Surprise!! $\bar{y}_1(t)$ and $\bar{y}_2(t)$ both are solutions of $\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \bar{y}$

check:

Note: $\bar{y}_1(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\bar{y}_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore \bar{y}_1(0)$ and $\bar{y}_2(0)$ are linearly independent

$\therefore$ The general solution for

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \bar{y}$$

can be written as $\bar{y}(t) = k_1 \bar{y}_1(t) + k_2 \bar{y}_2(t)$

Example: Solve the initial value problem

$$\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \bar{y} \quad \bar{y}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Linear Systems with Complex Eigenvalues

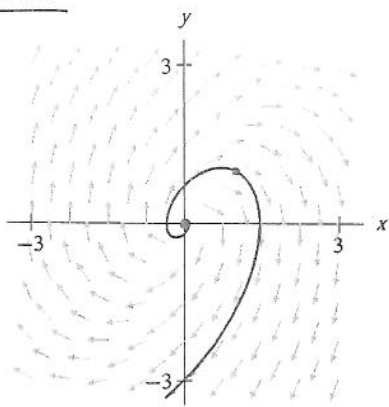
Given a linear system with complex eigenvalues $\lambda = \alpha \pm \beta i$, $\beta > 0$, the solutions spiral around the origin in the phase plane with a period $\frac{2\pi}{\beta}$

• If $\alpha < 0$, the solutions spiral towards the origin. The origin is called a spiral sink.

• If $\alpha > 0$, the solutions spiral away from the origin. The origin is called a spiral source.

• If $\alpha = 0$, the solutions are periodic. They return to exactly their initial condition in the phase plane and repeat the same closed curve over and over. The origin is called a center.

A spiral source



Equilibrium point is a
center

