Sec 3.4

Systems without straight-Line Solutions

Example 1 $\dfrac{d\bar{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\bar{Y} = A\bar{Y}$

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

Example 2 $\dfrac{d\bar{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}\bar{Y} = B\bar{Y}$

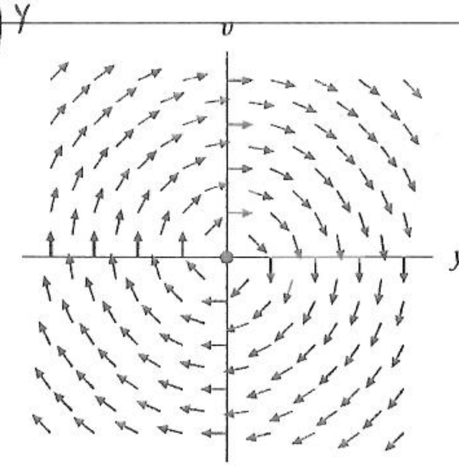
$$\det(B-\lambda I) = \det\begin{pmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{pmatrix} = 0$$

$$(-2-\lambda)(-2-\lambda) + 9 = 0$$

Example 1

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\bar{Y}$$

$$\begin{cases} \dfrac{dy}{dt} = v \\ \dfrac{dv}{dt} = -y \end{cases}$$



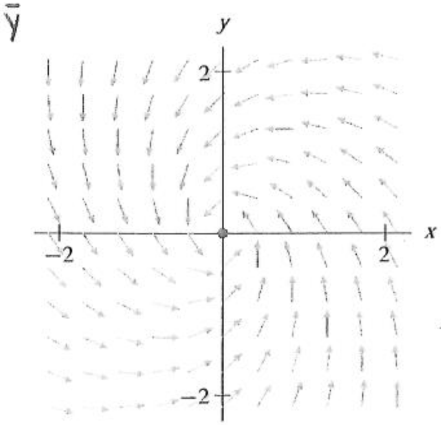
A Solution

$y(t) = \cos t$

$v(t) = -\sin t$

Example 2

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}\bar{Y}$$



A solution

$x(t) = -e^{-2t}\sin(3t)$

$y(t) = e^{-2t}\cos(3t)$

$B = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$ has eigenvalues

$\lambda_1 = -2+3i \qquad \lambda_2 = -2-3i$

Let's Find the corresponding eigenvectors

For $\lambda_1 = -2+3i$, we want vector $\bar{V}_1$ such that

$$B\bar{V}_1 = \lambda \bar{V}_1 \qquad \text{let } \bar{V}_1 = (x_1, y_1)$$

$$\begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = (-2+3i)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

<u>Note</u>: If λ is an eigenvalue of B and $\bar{V}$ is the corresponding eigenvector so that $\bar{B}\bar{V} = \lambda \bar{V}$ (where λ and $\bar{V}$ are complex) then $\bar{Y}(t) = e^{\lambda t}\bar{V}$ is a solution of

$$\frac{d\bar{Y}}{dt} = B\bar{Y}$$

We have shown that

$$\bar{y}(t) = e^{(-2+3i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} \qquad (*)$$

is a solution of $\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}\bar{y}$

<u>Problem</u>: We don't want complex valued solutions.

<u>Euler's Formula</u>: $e^{i\theta} = \cos\theta + i\sin\theta$

We can rewrite $\bar{y}(t)$ using Euler's Formula

$$\bar{y}(t) = e^{(-2+3i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{-2t}\cdot e^{3ti}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\bar{y}(t) = e^{(-2+3i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\bar{y}(t) = e^{-2t}\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + i\, e^{-2t}\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$$

Let $\bar{Y}_1(t) = \mathrm{Re}(\bar{Y}(t)) = e^{-2t}\begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix}$

$$Y_2(t) = \mathrm{Im}(\bar{Y}(t)) = e^{-2t}\begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}$$

Note: $\bar{Y}_1(t)$ and $\bar{Y}_2(t)$ are real valued functions

<u>Surprise</u>!! $\bar{Y}_1(t)$ and $\bar{Y}_2(t)$ both are solutions of $\frac{d\bar{y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}\bar{y}$

<u>check</u>:

Note: $\bar{Y}_1(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\bar{Y}_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore$ $\bar{Y}_1(0)$ and $\bar{Y}_2(0)$ are linearly independent

$\therefore$ The general solution for

$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \bar{Y}$$

can be written as $\bar{Y}(t) = k_1 \bar{Y}_1(t) + k_2 \bar{Y}_2(t)$

Example: Solve the initial value problem

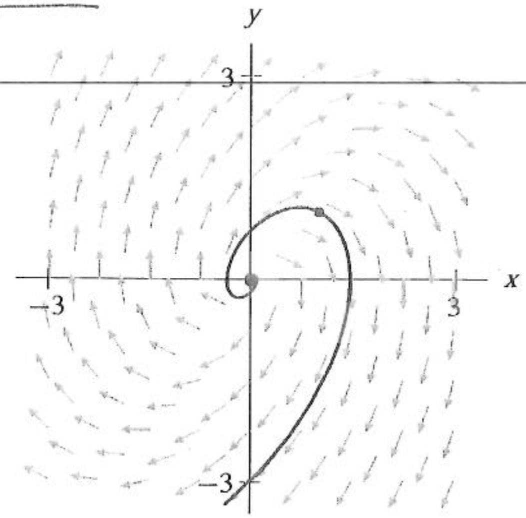
$$\frac{d\bar{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \bar{Y} \qquad \bar{Y}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Linear Systems with Complex Eigenvalues

Given a linear system with complex eigenvalues $\lambda = \alpha \pm \beta i$, $\beta > 0$, the solutions spiral around the origin in the phase plane with a period $\frac{2\pi}{\beta}$

- If $\alpha < 0$, the solutions spiral towards the origin. The origin is called a <u>spiral sink</u>.
- If $\alpha > 0$, the solutions spiral away from the origin. The origin is called a <u>spiral source</u>
- If $\alpha = 0$, the solutions are periodic. They return to exactly their initial condition in the phase plane and repeat the same closed curve over and over. The origin is called a <u>center</u>.

A spiral source



Equilibrium point is a center