Sec 3.3 Phase Portraits for Linear Systems with Real Eigenvalues

- If a linear system of the form $\frac{d\bar{Y}}{dt} = A\bar{Y}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real eigenvalues then $(0,0)$ is the only equilibrium point.
- Recall that the differential equations from chapter 1 of the form $\frac{dy}{dt} = f(y)$ had 3 possible types of equilibrium values <u>sinks</u>, <u>sources</u>, and <u>nodes</u>
- When working with linear systems $\frac{d\bar{Y}}{dt} = A\bar{Y}$ with 2 real eigenvalues the equilibrium point at the origin will be classified as either a <u>sink</u>, <u>source</u>, or <u>saddle</u>

<u>Saddles</u>

<u>Example 1</u> $\frac{d\bar{Y}}{dt} = A\bar{Y}$ $A = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$

Eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$

Corresponding eigenvectors $\bar{V}_1 = (1,0)$, $\bar{V}_2 = (0,1)$

General Solution

$$\bar{Y}(t) = k_1 e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

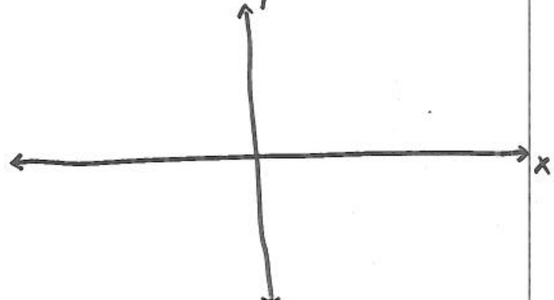
Special cases

$k_1 = 1$, $k_2 = 0$ $\bar{Y}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$k_1 = -1$, $k_2 = 0$

$k_1 = 0$, $k_2 = 1$

$k_1 = 0$, $k_2 = -1$



<u>Saddles - Example 2</u>

$\frac{d\bar{Y}}{dt} = B\bar{Y}$ $B = \begin{pmatrix} 8 & -11 \\ 6 & -9 \end{pmatrix}$

Eigenvalues $\lambda_1 = -3$ $\lambda_2 = 2$

Eigenvectors $\bar{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\bar{V}_2 = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$

General Solution

$\bar{Y}(t) = k_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} 11 \\ 6 \end{pmatrix}$

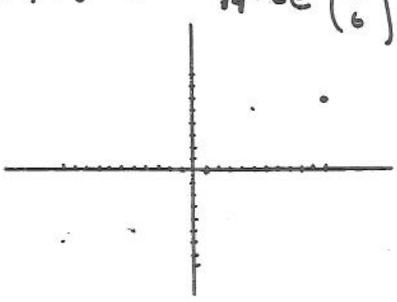
<u>Straight Line Solutions</u>

$k_1 = 1$, $k_2 = 0$ $\bar{Y}_1 = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$k_1 = -1$, $k_2 = 0$ $\bar{Y}_2 = -e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$k_1 = 0$, $k_2 = 1$ $\bar{Y}_3 = e^{2t}\begin{pmatrix} 11 \\ 6 \end{pmatrix}$

$k_1 = 0$, $k_2 = -1$ $\bar{Y}_4 = -e^{2t}\begin{pmatrix} 11 \\ 6 \end{pmatrix}$



<u>Sinks</u> Example 1

$\frac{d\bar{Y}}{dt} = C\bar{Y}$ $C = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}\bar{Y}$

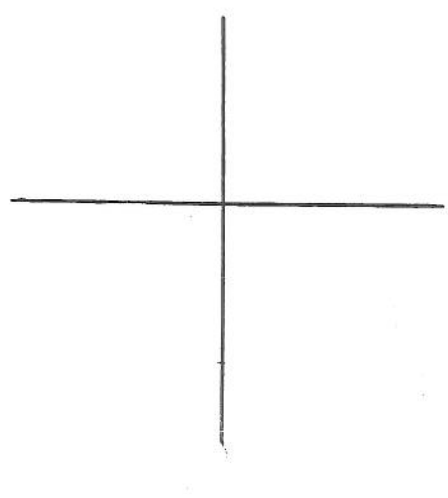
General Solution

$$\bar{Y}(t) = k_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>Straight Line Solutions</u>

$\bar{Y}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\bar{Y}_3(t) = e^{-4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$Y_2(t) = -e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\bar{Y}_4(t) = -e^{-4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Sinks - Example 2

$$\frac{d\bar{Y}}{dt} = D\bar{Y} \qquad D = \begin{pmatrix} -2 & -2 \\ -1 & 3 \end{pmatrix}$$

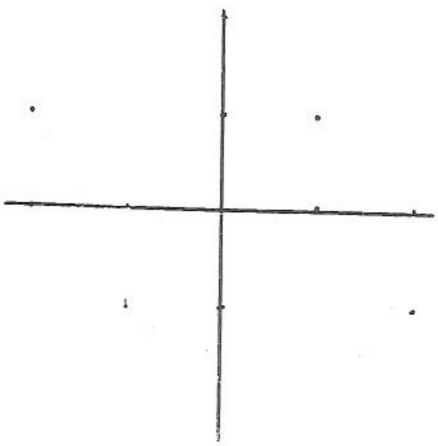
General Solution

$$\bar{Y}(t) = k_1 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Straight Line Solutions

$\bar{Y}_1(t) = e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\bar{Y}_3(t) = e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\bar{Y}_2(t) = -e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\bar{Y}_4(t) = -e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$



Sources

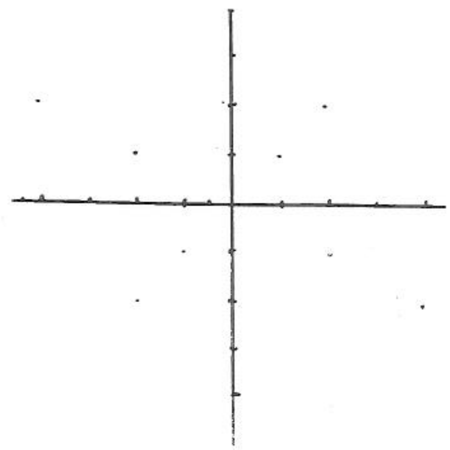
$$\frac{d\bar{Y}}{dt} = E\bar{Y} \qquad E = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

General Solution $\bar{Y}(t) = k_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Straight Line Solutions

$\bar{Y}_1(t) = e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\bar{Y}_3(t) = e^{t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\bar{Y}_2(t) = -e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\bar{Y}_4(t) = -e^{t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$



Stable and Unstable Equilibrium Points

Three types of equilibrium points

- Saddles: If $\lambda_1 < 0 < \lambda_2$ then the origin is a saddle point. There are two lines in the phase portrait that correspond to straight-line solutions. Solutions along one line tend to $(0,0)$ as t increases, and solutions on the other line tend away from $(0,0)$ as t increases.
- Sinks: If $\lambda_1 < \lambda_2 < 0$. All solutions tend to $(0,0)$ as time increases and most tend to $(0,0)$ in the direction of λ_2 eigenvector
- Sources: If $0 < \lambda_2 < \lambda_1$. All solutions except the equilibrium solution go to infinity as t increases, and most solution curves leave the origin in the direction of the λ_2 eigenvector.