

Sec 3.3 Phase Portraits for Linear Systems

with Real Eigenvalues

- If a linear system of the form $\frac{d\bar{y}}{dt} = A\bar{y}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real eigenvalues then $(0,0)$ is the only equilibrium point.
- Recall that the differential equations from chapter 1 of the form $\frac{dy}{dt} = f(y)$ had 3 possible types of equilibrium values sinks, sources, and nodes
- When working with linear systems $\frac{d\bar{y}}{dt} = A\bar{y}$ with 2 real eigenvalues the equilibrium point at the origin will be classified as either a sink, source, or saddle

Saddles

Example 1 $\frac{d\bar{y}}{dt} = A\bar{y}$ $A = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$

Eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$

corresponding eigenvectors $\bar{V}_1 = (1, 0)$, $\bar{V}_2 = (0, 1)$

General Solution

$$\bar{Y}(t) = k_1 e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Special cases

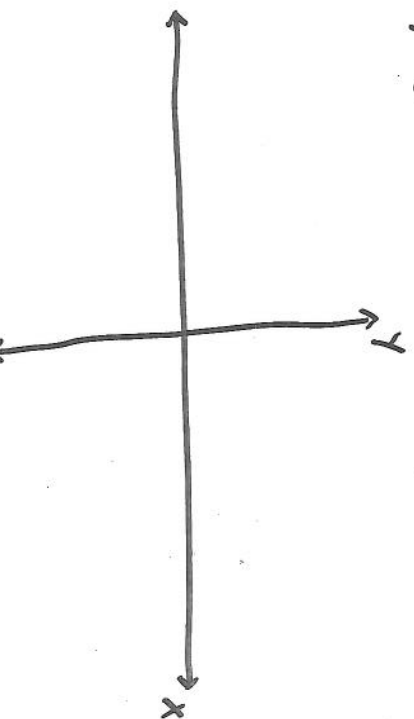
$$k_1 = 1, k_2 = 0$$

$$\bar{Y}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$k_1 = -1, k_2 = 0$$

$$k_1 = 0, k_2 = 1$$

$$k_1 = 0, k_2 = -1$$



Saddles - Example 2

$$\frac{d\bar{Y}}{dt} = B\bar{Y} \quad B = \begin{pmatrix} 8 & -11 \\ 6 & -9 \end{pmatrix}$$

Eigenvalues $\lambda_1 = -3$ $\lambda_2 = 2$

Eigenvectors $\bar{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\bar{V}_2 = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$

General Solution

$$\bar{Y}(t) = K_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + K_2 e^{2t} \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

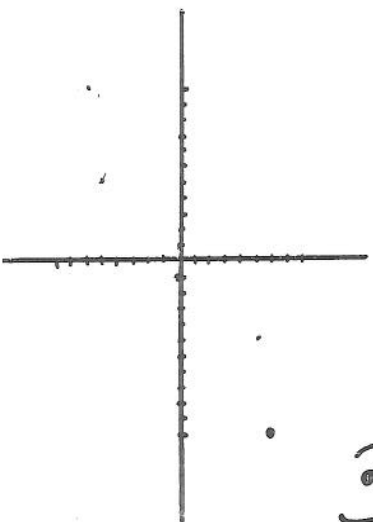
Straight Line Solutions

$$K_1 = 1, K_2 = 0 \quad \bar{Y}_1 = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$K_1 = -1, K_2 = 0 \quad \bar{Y}_2 = -e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$K_1 = 0, K_2 = 1 \quad \bar{Y}_3 = e^{2t} \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

$$K_1 = 0, K_2 = -1 \quad \bar{Y}_4 = -e^{2t} \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$



Sinks Example 1

$$\frac{d\bar{Y}}{dt} = C\bar{Y} \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \bar{Y}$$

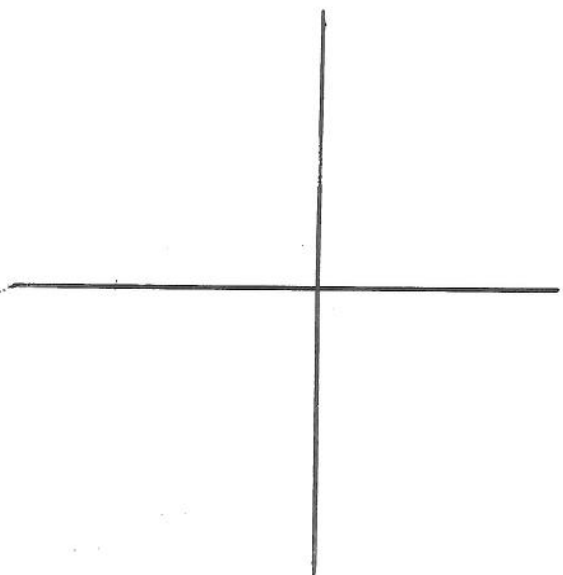
General Solution

$$\bar{Y}(t) = K_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Straight Line Solutions

$$\bar{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \bar{Y}_3(t) = e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Y_2(t) = -e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \bar{Y}_4(t) = -e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Sinks - Example 2

$$\frac{d\bar{y}}{dt} = D\bar{y} \quad D = \begin{pmatrix} -2 & -2 \\ -1 & 3 \end{pmatrix}$$

General Solution

$$\bar{y}(t) = k_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

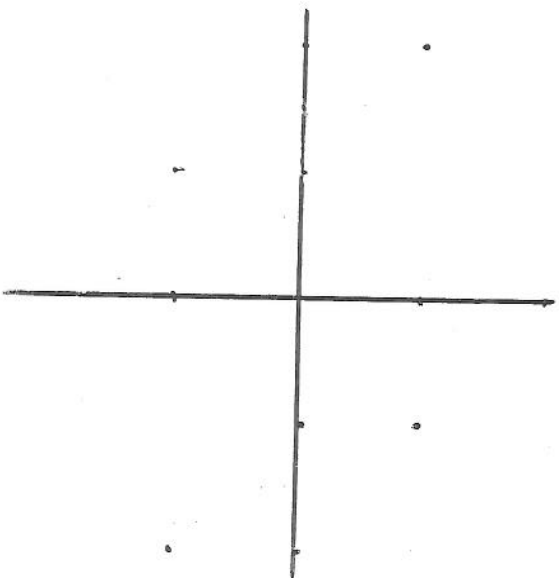
Straight Line Solutions

$$\bar{y}_1(t) = e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{y}_3(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\bar{y}_2(t) = -e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{y}_4(t) = -e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$



Sources

$$\frac{d\bar{y}}{dt} = E\bar{y} \quad E = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

General Solution

$$\bar{y}(t) = k_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

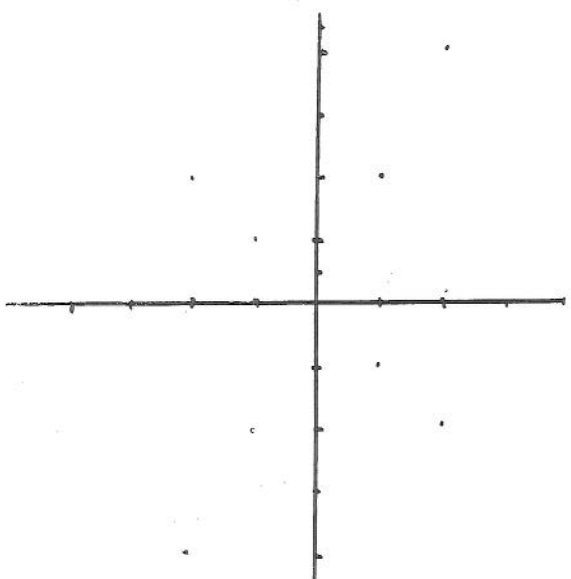
Straight Line Solutions

$$\bar{y}_1(t) = e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{y}_3(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\bar{y}_2(t) = -e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{y}_4(t) = -e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$



Stable and Unstable Equilibrium Points

Three types of equilibrium points

- Saddles If $\lambda_1 < 0 < \lambda_2$ then the origin is a saddle point. There are two lines in the phase portrait that correspond to straight-line solutions. Solutions along one line tend to $(0,0)$ as t increases, and solutions on the other line tend away from $(0,0)$ as t increases.
- Sinks: If $\lambda_1 < \lambda_2 < 0$. All solutions tend to $(0,0)$ as time increases and most tend to $(0,0)$ in the direction of λ_2 eigenvector.
- Sources: If $0 < \lambda_2 < \lambda_1$. All solutions except the equilibrium solution go to infinity as t increases, and most solution curves leave the origin in the direction of the λ_1 eigenvector.