Blue

Prof D

MA 226 Section B – Exam 1b

Question Number	Possible Points	Student Score
1	15	
2a	10	
2b	10	
2c	10	
2d	10	
3	12	
4	15	
5	10	
6	8	
Total Points	100	

You must show your work to receive full credit

Discussion Sections:

B2: Wednesday 9-10

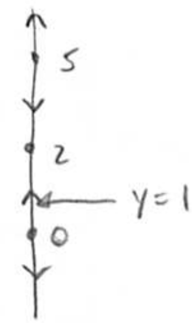
B3: Wednesday 2-3

1. Short Answer (15 pts)

a) Given the differential equation : $\frac{dy}{dt} = 3y(y-2)(y-5)$. Let $y_1(t)$ be a solution that satisfies the initial condition $y_1(0) = 1$. Evaluate

$-$ 0 $+$ 2 $-$ 5 $+$

$\lim_{t \to \infty} y_1(t) =$ 2 $\qquad\qquad$ $\lim_{t \to -\infty} y_1(t) =$ 0



b) Find the equilibrium solutions of the differential equation $\frac{dy}{dt} = y^3 + 4y^2 + 3y$ $= y(y^2+4y+3)$

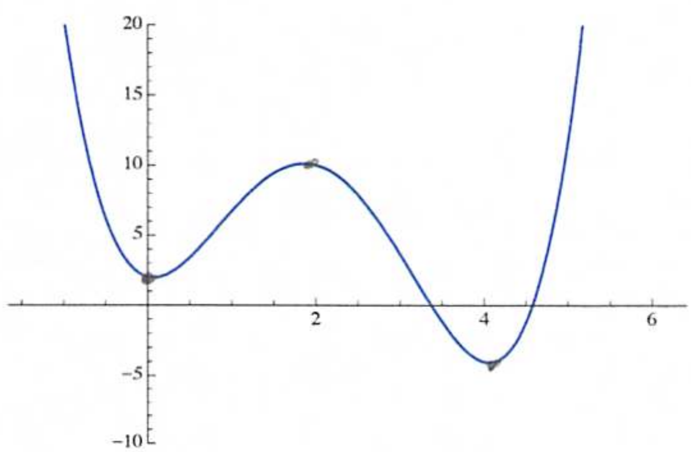
$= y(y+3)(y+1)$

$y = 0$

$y = -3$

$y = -1$

c) Given the one parameter family of differential equations $\frac{dy}{dt} = f(y) + \alpha$ where $f(y)$ is given by the graph below, identify the bifurcation value(s). You **DO NOT** need to draw a bifurcation diagram just identify the bifurcation values. Note that the critical points on the graph of $f(y)$ occur at (0,2) , (2,10) and (4,-4)



$\alpha = -2$

$\alpha = -10$

$\alpha = 4$

2. Solving differential equations and initial value problems

a) (10 pts) Find the general solution of the equation: $\dfrac{dy}{dt} = \dfrac{3y(5-y)}{10}$

$$\frac{dy}{y(5-y)} = \frac{3}{10}dt$$

$$\frac{1}{y(5-y)} = \frac{A}{y} + \frac{B}{5-y}$$

$$1 = 5A - Ay + By$$

3pts

$$5A = 1 \Rightarrow \boxed{A = \tfrac{1}{5}}$$

$$(-A+B) = 0 \Rightarrow \boxed{B = \tfrac{1}{5}}$$

$$\int \frac{dy}{y(5-y)} = \int \frac{3}{10}dt$$

$$\frac{1}{5}\int\left(\frac{1}{y} + \frac{1}{5-y}\right)dy = \int \frac{3}{10}dt$$

$$\ln|y| - \ln|5-y| = \frac{3}{2}t + c$$

$$\ln\left|\frac{y}{5-y}\right| = \frac{3}{2}t + c$$

3 pts $\boxed{\dfrac{y}{5-y} = ce^{\frac{3}{2}t}}$

General Solution

$$y(t) = \begin{cases} \dfrac{5ce^{3/2t}}{1+ce^{3/2t}} & y \neq 5 \\ 5 & \end{cases}$$

1 pt

$$y = (5-y)ce^{\frac{3}{2}t}$$

$$y = 5ce^{\frac{3}{2}t} - yce^{\frac{3}{2}t}$$

$$y + yce^{3/2t} = 5ce^{3/2t}$$

$$y\left(1 + ce^{3/2t}\right) = 5ce^{3/2t}$$

3 pts $\boxed{y(t) = \dfrac{5ce^{3/2t}}{1+ce^{3/2t}}}$

b) (10 pts) Solve the initial value problem $\frac{dy}{dt} + 3y = 12\sin(3t) + 9t$ with $y(0) = 0$

① Associated Homogeneous problem

$$\frac{dy}{dt} = -3y \qquad \boxed{y_h(t) = ke^{-3t}}$$

2 pts

② Particular solution for

$$\frac{dy}{dt} + 3y = 12\sin 3t \qquad (*)$$

$$y_{p_1}(t) = A\sin(3t) + B\cos(3t)$$

$$y_{p_1}'(t) = 3A\cos(3t) - 3B\sin(3t)$$

Plug into $(*)$

$$3A\cos 3t - 3B\sin 3t + 3(A\sin 3t + B\cos 3t) = 12\sin 3t$$

$$(3A + 3B)\cos(3t) + (3A - 3B)\sin(3t) = 12\sin 3t$$

$$3A + 3B = 0 \Rightarrow B = -A$$

$$3A - 3B = 12$$

$$3A + 3A = 6A = 12 \Rightarrow A = 2$$
$$B = -2$$

3 pts

$$\boxed{y_{p_1}(t) = 2\sin(3t) - 2\cos(3t)}$$

③ Particular solution for

$$\frac{dy}{dt} + 3y = 9t \qquad (**)$$

3 pts $\boxed{y_{p_2}(t) = 3t - 1}$

$y_{p_2}(t) = At + B$, plug into $(**)$

$$A + 3(At + B) = 9t$$

$$3At + (A + 3B) = 9t$$

$$A = 3, \quad B = -\frac{A}{3} = -1$$

General Solution

$$y_G(t) = ke^{-3t} + 2\sin(3t) - 2\cos(3t) + 3t - 1$$

1 pts

$$y(0) = 0 = k - 2 - 1 \Rightarrow k = 3$$

$$\boxed{y(t) = 3e^{-3t} + 2\sin(3t) - 2\cos(3t) + 3t\cancel{+}1}$$

1 pt ~~pts~~

c) (10 pts) Solve the initial value problem $\frac{dy}{dt} = -2t \cdot y + 4e^{-t^2}$ with $y(0) = 3$

$$\frac{dy}{dt} + 2t \cdot y = 4e^{-t^2}$$

$$\boxed{\mu(t) = e^{\int 2t\,dt} = e^{t^2}}$$ 4 pts

$$e^{t^2} \cdot \left(\frac{dy}{dt} + 2ty \right) = e^{t^2} \cdot 4e^{-t^2}$$

$$\frac{d}{dt}\left(e^{t^2} \cdot y \right) = 4$$

$$\int \frac{d}{dt}\left(e^{t^2} \cdot y \right) dt = \int 4\,dt$$

$$e^{t^2} \cdot y = 4t + c$$

$$\boxed{y(t) = \frac{4t + c}{e^{t^2}}}$$ 4 pts

$$y(0) = 3 \Rightarrow 3 = \frac{c}{1} \quad c = 3$$ 1 pt

$$y(t) = \frac{4t+3}{e^{t^2}}$$ 1 pt

(*)

d) (10 pts) Solve the initial value problem with $\frac{dy}{dt} = \frac{y}{2} + 4e^{t/2}$ and $y(0) = 4$

$\boxed{\text{Step 1: } y_h(t) = ke^{\frac{1}{2}t}}$ 2 pts

Step 2 $y_p(t) = Ate^{\frac{1}{2}t}$ 3 pts

plug into (*)

$$Ae^{\frac{1}{2}t} + \cancel{\tfrac{1}{2}Ate^{\frac{1}{2}t}} = \cancel{\tfrac{1}{2}Ate^{\frac{1}{2}t}} + 4e^{\frac{t}{2}}$$

$\boxed{A = 4}$ 2 pts

$$\boxed{y(t) = ke^{\frac{1}{2}t} + 4te^{\frac{1}{2}t}}$$

$y(0) = 4 = K$ 1 pt

$$\boxed{y(t) = 4e^{\frac{1}{2}t} + 4te^{\frac{1}{2}t}}$$

2 pts

3. (12 pts) Matching slope fields

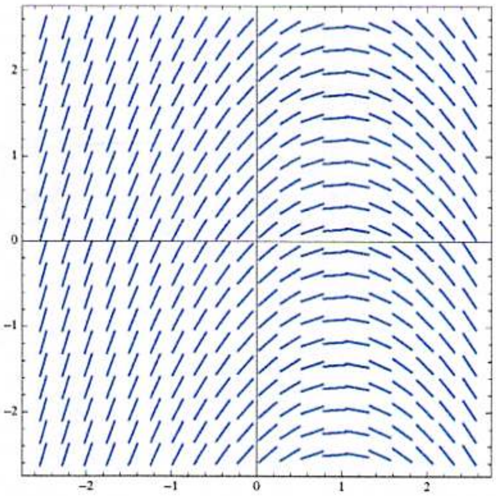
(i) $\frac{dy}{dt} = t-1$ (ii) $\frac{dy}{dt} = 1-y^2$ (iii) $\frac{dy}{dt} = y-t^2$ V (iv) $\frac{dy}{dt} = 1-t$

viii (iii) (ii)

(v) $\frac{dy}{dt} = 1-y$ (vi) $\frac{dy}{dt} = y+t^2$ (vii) $\frac{dy}{dt} = ty-t$ (viii) $\frac{dy}{dt} = y^2-1$

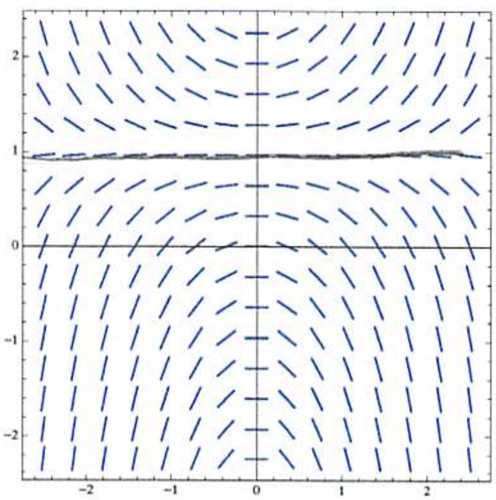
Slope Field A equation: ~~(i)~~ or (iv) Reason: at $t=0$ slope is positive

depends only on t



Slope Field B equation: (vii) Reason Equilibrium value $y=1$

must depend on both t and y

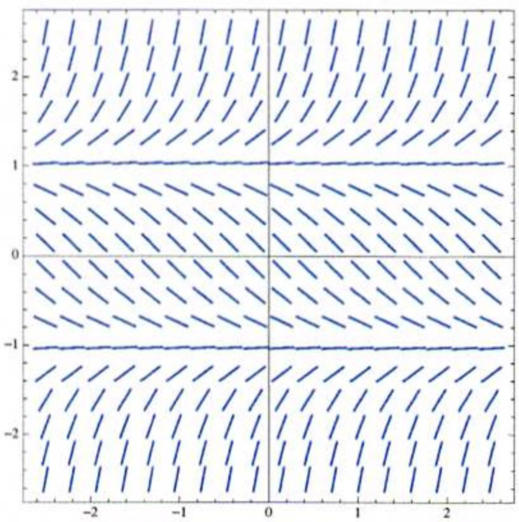


(i) $\frac{dy}{dt} = t - 1$ (ii) $\frac{dy}{dt} = 1 - y^2$ (iii) $\frac{dy}{dt} = y - t^2$ (iv) $\frac{dy}{dt} = 1 - t$

(v) $\frac{dy}{dt} = 1 - y$ (vi) $\frac{dy}{dt} = y + t^2$ (vii) $\frac{dy}{dt} = ty - t$ (viii) $\frac{dy}{dt} = y^2 - 1$

Slope Field C equation: viii

Reason



must be (ii) (v) or (vii)

2 equilibrium points $y=1$, $y=-1$

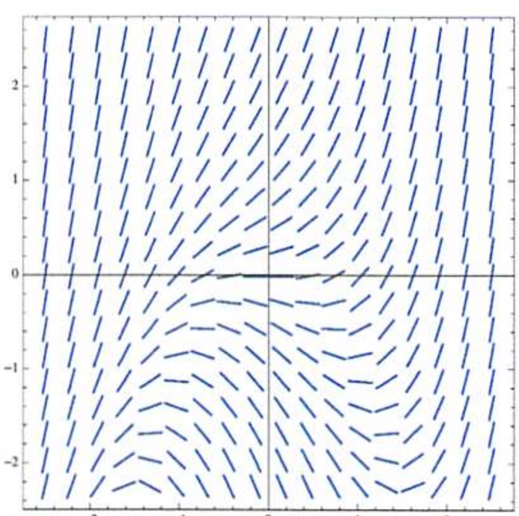
can't be (v) only one equilibrium pt.

Must be viii because for $-1<y<1$

slope is negative

Slope Field D equation: Vi

Reason



B

must be (vi), (iii), or ~~(vii)~~

when $y=0$ slope is positive

must be (vi)

4. (15 pts) Consider a large vat containing sugar water that is to be made into soft drinks.

- Initially the vat contains only pure water.

-The The vat contains 100 gals of liquid. Moreover, the amount flowing in is the same as the amount flowing out, so there are always 100 gallons in the vat.

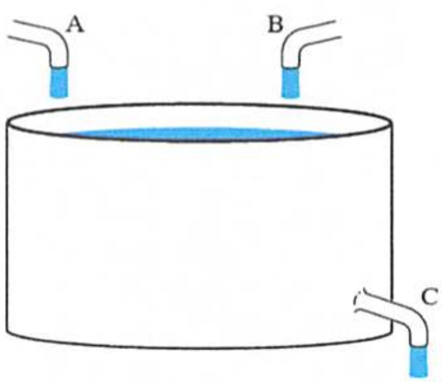
- The vat is kept well mixed, so that the sugar concentration is uniform throughout the vat.

-Sugar water containing 4 tablespoons per gallon enters the vat through pipe A at a rate of 3 gallons per minute.

-Sugar water containing 6 tablespoons per gallon enters the vat through pipe B at at rate of 2 gallon per minute.

-Sugar water leaves the vat through pipe C at a rate of 5 gallons per minute.

a.) (4 pts) : Write the initial value problem that describes the amount of sugar in the vat as a function of time.



Let $S(t)$ = Tsp of sugar at time t

$S(0) = 0$

$$\frac{dS}{dt} = 4\,\frac{\text{Tsp}}{\text{gal}} \cdot 3\,\frac{\text{gal}}{\text{min}} + 6\,\frac{\text{Tsp}}{\text{gal}} \cdot 2\,\frac{\text{gal}}{\text{min}} - \frac{S(t)}{100}\,\frac{\text{Tsp}}{\text{gal}} \cdot 5\,\frac{\text{gal}}{\text{min}}$$

$$\frac{dS}{dt} = 12 + 12 - \frac{S}{20} = 24 - \frac{S}{20}$$

IVP. $\begin{cases} \dfrac{dS}{dt} = \dfrac{480 - S}{20} \\ S(0) = 0 \end{cases}$

3pts

1pt

b.) (7 pts) : Solve the initial value problem Separable

$$\frac{dS}{dt} = \frac{480 - S}{20}$$

$$\frac{dS}{480 - S} = \frac{dt}{20}$$

$$\int \frac{dS}{480 - S} = \int \frac{1}{20} dt$$

$$-\ln|480 - S| = \frac{1}{20}t + c$$

$$\ln|400 - S| = -\frac{1}{20}t + c$$

$$480 - S = ce^{-\frac{1}{20}t}$$

$$\boxed{S(t) = 480 + ce^{-\frac{1}{20}t}}$$ 5pts

$$S(0) = 0 = 480 + c \Rightarrow \boxed{c = -480}$$ 2pts

$$S(t) = 480\left(1 - e^{-\frac{1}{20}t}\right)$$

c.) (2 pts): How much sugar will be in the tank after 30 minutes? $S(30) \approx 372.9$ Tsp

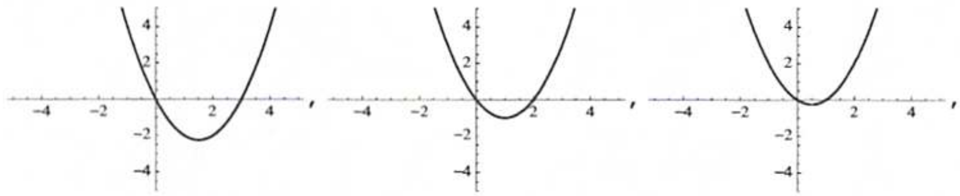
d.) (2 pts) : What is the limit for the amount of sugar in the vat for very large values of time?

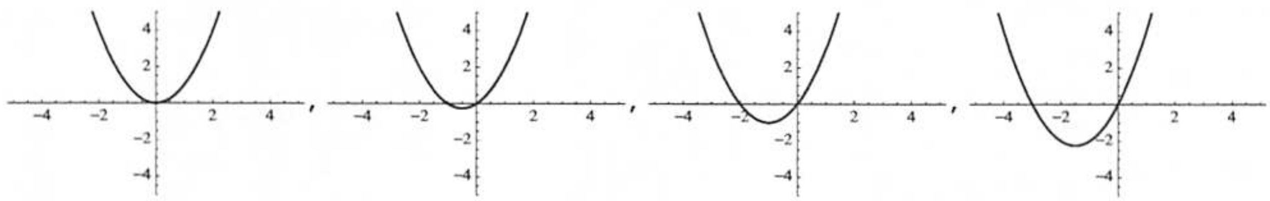
$$\lim_{t \to \infty} S(t) = 480$$

5. (10 pts) Given the one parameter family of differential equations:

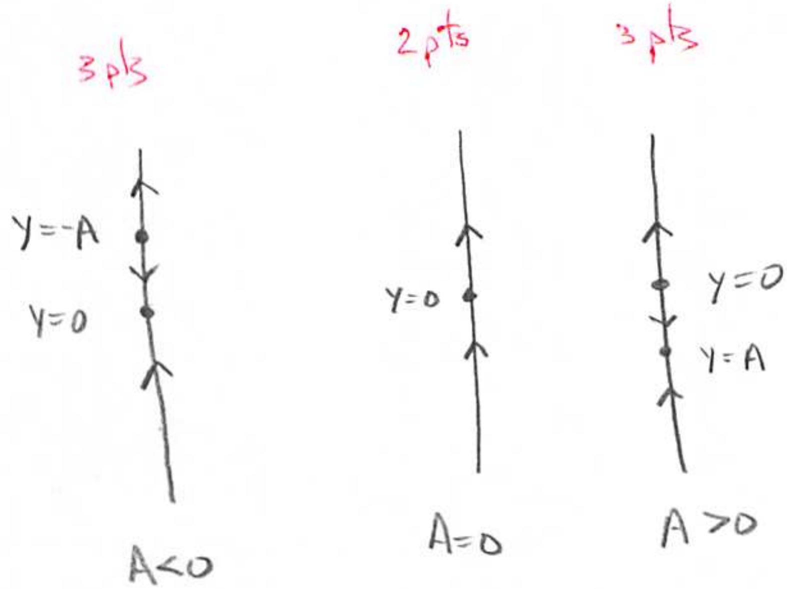
$\frac{dy}{dt} = Ay + y^2$ and a parameter study for values of the parameter A ranging from A = -3 to 3 create a bifurcation diagram and indicate the bifurcation value(s).

A = -3 A = -2 A=-1





A = 0 A = 1 A = 2 A = 3



$\boxed{\text{A=0 is The bifurcation value}}$ 2 pts

6. (8 pts) Use Euler's Method with a step size of 0.25 to approximate the solution of the initial value problem $\frac{dy}{dt} = y^2 - 2t$ and $y(0) = 1$ over the interval $0 \le t \le .75$.

Create a table that shows how you computed the approximate y values for values for t= .25, .5, and .75

K	t_K	y_K	$f(t_K, y_K) = y_K^2 - 2t_K$	$y_{K+1} = y_K + f(t_K, y_K) \cdot (.25)$
0	0	1	1 1pt	$1 + 1(.25) = 1.25$ 1pt
1	.25	1.25 1pt	1.0625 1pt	1.515625 1pt
2	.5	1.515625 1pt	1.297119 1pt	1.839905 1pt
3	.75	1.839905		