Prof D

# MA 226 Section B – Exam 1a

| Question Number | Possible Points | Student Score |
|---|---|---|
| 1 | 15 | |
| 2a | 10 | |
| 2b | 10 | |
| 2c | 10 | |
| 2d | 10 | |
| 3 | 12 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 8 | |
| Total Points | 100 | |

## You must show your work to receive full credit

Discussion Sections:

B2: Wednesday 9-10

B3: Wednesday 2-3

Prof D

1. Short Answer (15 pts)

a) Given the differential equation : $\frac{dy}{dt} = 3(y-1)(y-5)(y-7)$. Let $y_1(t)$ be a solution that satisfies the initial condition $y_1(0) = 3$. Evaluate

$\lim_{t \to \infty} y_1(t) =$ 5 $\qquad$ $\lim_{t \to -\infty} y_1(t) =$ 1

− + − +

1 5 7


b) Find the equilibrium solutions of the differential equation $\frac{dy}{dt} = y^3 + 2y^2 - 8y$ $= y(y^2+2y-8)$

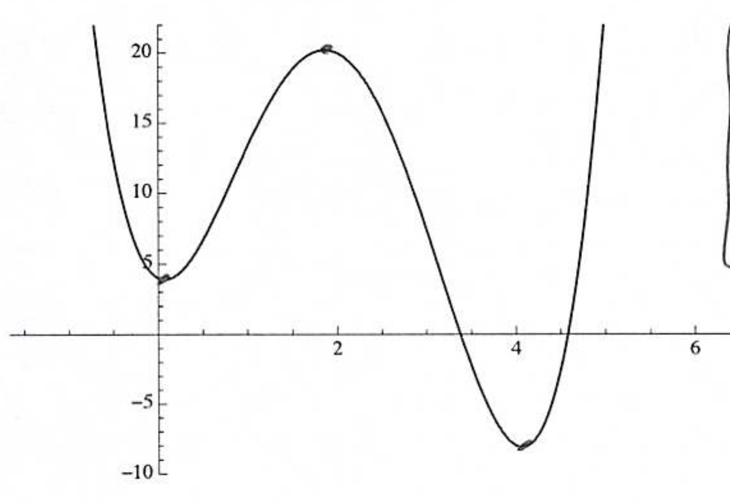
$= y(y+4)(y-2)$

Equilibrium solution $y(t) = 0$

$y(t) = -4$

$y(t) = 2$

c) Given the one parameter family of differential equations $\frac{dy}{dt} = f(y) + \alpha$ where $f(y)$ is given by the graph below, identify the bifurcation value(s). You **DO NOT** need to draw a bifurcation diagram just identify the bifurcation values. Note that the critical points on the graph of $f(y)$ occur at (0,4) , (2,20) and (4,-8)




$\boxed{\begin{array}{l} \alpha = -4 \\ \alpha = -20 \\ \alpha = 8 \end{array}}$

2. Solving differential equations and initial value problems

a) (10 pts) Find the general solution of the equation: $\frac{dy}{dt} = \frac{5y(4-y)}{8}$

Separable

$$\frac{dy}{y(4-y)} = \frac{5}{8}dt$$

$$\frac{1}{4}\int \frac{1}{y} + \frac{1}{4-y}\,dy = \int \frac{5}{8}dt$$

$$\frac{1}{4}\left(\ln|y| - \ln|4-y|\right) = \frac{5}{8}t + c$$

$$\ln\left|\frac{y}{4-y}\right| = \frac{5}{2}t + c$$

$$\boxed{\frac{y}{4-y} = ce^{\frac{5}{2}t}}$$ 3 pts

$$y = (4-y)\,ce^{\frac{5}{2}t}$$

$$y = 4ce^{5/2t} - yce^{5/2t}$$

$$y(1+ce^{5/2t}) = 4ce^{5t/2}$$

$$\boxed{y(t) = \frac{4ce^{5/2t}}{1+ce^{5/2t}}}$$ 3 pts

Partial Fractions

$$\frac{1}{y(4-y)} = \frac{A}{y} + \frac{B}{4-y}$$ 3 pts

$$= \frac{4A - Ay + By}{y(4-y)}$$

$$\Rightarrow 4A = 1 \qquad \Rightarrow \boxed{A = \tfrac{1}{4} \quad B = \tfrac{1}{4}}$$

$$-A + B = 0$$

General Solution

$$y(t) = \begin{cases} \frac{4ce^{5/2t}}{1+ce^{5/2t}}, & y \neq 4 \\ 4 \end{cases}$$

1 pt

b) (10 pts) Solve the initial value problem $\frac{dy}{dt}+3y=12\sin(3t)+9t$ with $y(0)=0$

Step 1: Associated homogeneous problem

$\frac{dy}{dt} = -3y$ $\boxed{y_h(t) = Ke^{-3t}}$ 2 pts

Step 2: $y_{p_1}(t)$ for $\frac{dy}{dt} + 3y = 12\sin(3t)$ (*)

Let $y_{p_1}(t) = A\sin(3t) + B\cos(3t)$ plug into (*)

$3A\cos(3t) - 3B\sin(3t) + 3A\sin 3t + 3B\cos 3t = 12\sin 3t$

$(3A+3B)\cos(3t) + (3A-3B)\sin(3t) = 12\sin 3t$

$3A+3B=0 \Rightarrow B=-A$

$3A-3B=12 \Rightarrow 6A=12$

$\boxed{A=2}$ $\boxed{B=-2}$

$\boxed{y_{p_1}(t) = 2\sin(3t) - 2\cos(3t)}$ 3 pts

Step 3: $y_{p_2}(t)$ for $\frac{dy}{dt} + 3y = 9t$ (**)

Let $y_{p_2}(t) = At + B$, plug into (**)

$A + 3At + 3B = 9t$

$\Rightarrow 3A = 9$ $\boxed{A=3}$

$A+3B=0 \Rightarrow$ $\boxed{B = -\frac{A}{3} = -1}$

$\boxed{y_{p_2}(t) = 3t-1}$ 3 pts

Solution of IVP

when $y(0)=0$

$0 = K - 2 - 1$

$\Rightarrow K = 3$ 1 pt

$\boxed{y(t) = 3e^{-3t} + 2\sin(3t) - 2\cos 3t + 3t - 1}$

General Solution: $y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t)$

$y(t) = Ke^{-3t} + 2\sin 3(t) - 2\cos(3t) + 3t - 1$ 1 pt

c) (10 pts) Solve the initial value problem $\frac{dy}{dt} = -2t \cdot y + 5e^{-t^2}$ with $y(0) = 6$

$t^2$

Integrating Factor

$$\frac{dy}{dt} + 2t \cdot y = 5e^{-t^2}$$

$$\boxed{\mu(t) = e^{\int 2t\,dt} = e^{t^2}}$$ 4 pts

$$e^{t^2}\left(\frac{dy}{dt} + 2t\,y\right) = e^{t^2} \cdot 5e^{-t^2}$$

$$\frac{d}{dt}\left(e^{t^2} \cdot y\right) = 5$$

$$\int \frac{d}{dt}\left(e^{t^2} \cdot y\right) dt = \int 5\,dt$$

$$e^{t^2} \cdot y = 5t + c$$

4 pts $\boxed{y(t) = \frac{5t + c}{e^{t^2}}}$ general solution

Solution of IVP

$$y(0) = 6 = \frac{c}{1} \Rightarrow c = 6$$ 1 pt

$$\boxed{y(t) = \frac{5t + 6}{e^{t^2}}}$$ 1 pt

$\frac{t}{3}$

d) (10 pts) Solve the initial value problem with $\frac{dy}{dt} = \frac{y}{3} + 5e^{\frac{t}{3}}$ and $y(0) = 2$

Step 1 Associated homogeneous problem

$$\frac{dy}{dt} = \frac{1}{3}y \qquad \boxed{y_h(t) = ke^{\frac{1}{3}t}}$$ 2pts

Step 2 $\boxed{y_p(t) = Ate^{\frac{1}{3}t}}$ 3pts

Plug into eqn and solve for A

$$Ae^{\frac{1}{3}t} + \cancel{\frac{1}{3}Ate^{\frac{1}{3}t}} = \cancel{\frac{1}{3}Ate^{\frac{1}{3}t}} + 5e^{t/3}$$

$$\boxed{A = 5}$$ 2pts

Step 3: general solution

$$y(t) = ke^{\frac{1}{3}t} + 5te^{\frac{t}{3}}$$

$y(0) = 2 \Rightarrow k = 2$ 1pt

Solution of IVP

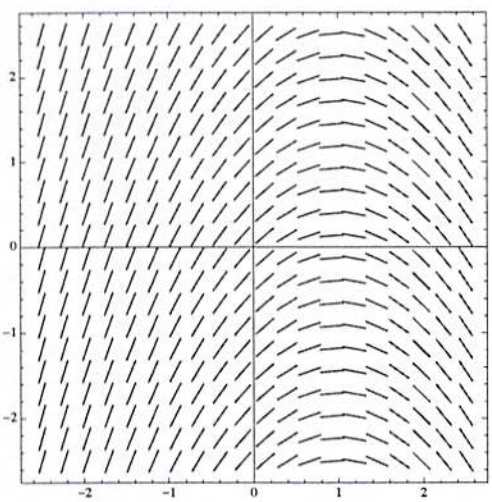
$$\boxed{y(t) = 2e^{\frac{1}{3}t} + 5te^{t/3}}$$ 2pts

3. (12 pts) Matching slope fields

(i) $\frac{dy}{dt} = t - 1$ (ii) $\frac{dy}{dt} = y^2 - 1$ (iii) $\frac{dy}{dt} = ty - t$ (iv) $\frac{dy}{dt} = 1 - y^2$

(v) $\frac{dy}{dt} = 1 - t$ (vi) $\frac{dy}{dt} = 1 - y$ (vii) $\frac{dy}{dt} = y - t^2$ (viii) $\frac{dy}{dt} = y + t^2$

Slope Field A equation: v Reason:

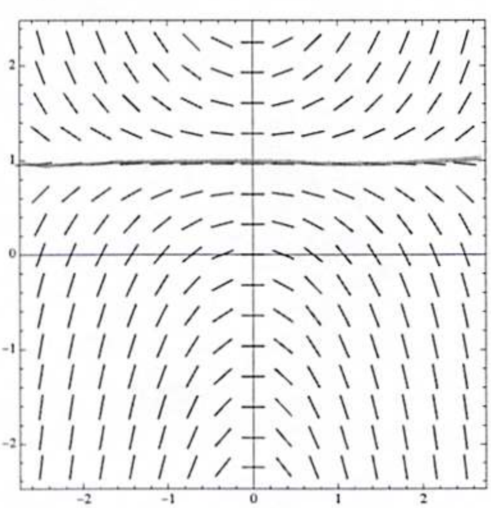


either ~~(i)~~ or (v)

t=0 slope is always positive

so it can't be (i)

Slope Field B equation: iii Reason



either (iii), ~~(vii)~~ or ~~(viii)~~

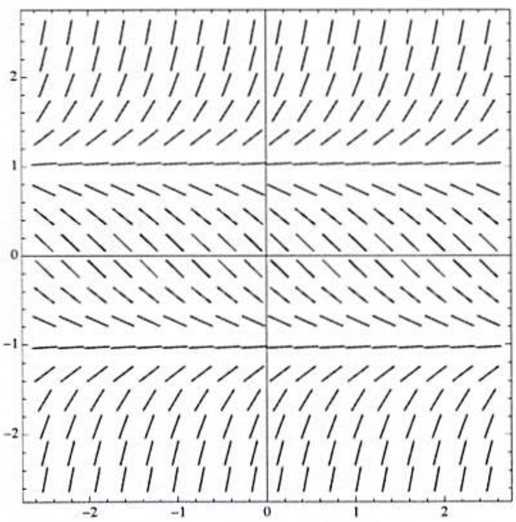
one equilibrium value y = 1

must be (iii)

(i) $\frac{dy}{dt} = t - 1$ (ii) $\frac{dy}{dt} = y^2 - 1$ (iii) $\frac{dy}{dt} = ty - t$ (iv) $\frac{dy}{dt} = 1 - y^2$

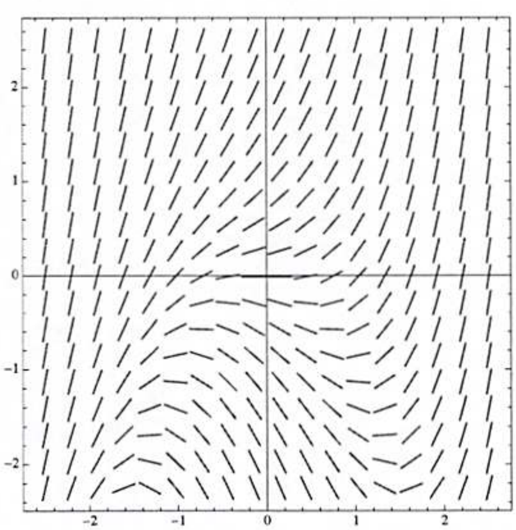
(v) $\frac{dy}{dt} = 1 - t$ (vi) $\frac{dy}{dt} = 1 - y$ (vii) $\frac{dy}{dt} = y - t^2$ (viii) $\frac{dy}{dt} = y + t^2$

Slope Field C equation: ii Reason



can only be (ii) (iv) or (vi)
2 equilibrium values 1 and -1
can't be (vi)
for $-1 < y < 1$ slope is negative
must be (ii)

Slope Field D equation: viii Reason



slope field B

must be ~~(iii)~~ (vii) (viii)
when $y=0$ slopes are all positive
must be (viii)

4. (15 pts) Consider a large vat containing sugar water that is to be made into soft drinks.

- Initially the vat contains only pure water.

-The The vat contains 100 gals of liquid. Moreover, the amount flowing in is the same as the amount flowing out, so there are always 100 gallons in the vat.

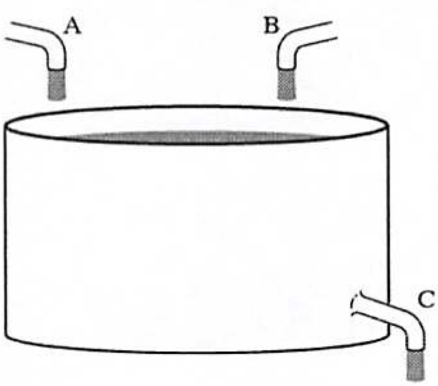
- The vat is kept well mixed, so that the sugar concentration is uniform throughout the vat.

-Sugar water containing 6 tablespoons per gallon enters the vat through pipe A at a rate of 3 gallons per minute.

-Sugar water containing 7 tablespoons per gallon enters the vat through pipe B at at rate of 2 gallon per minute.

-Sugar water leaves the vat through pipe C at a rate of 5 gallons per minute.

a.) (4 pts) : Write the initial value problem that describes the amount of sugar in the vat as a function of time.




Let $S(t)$ = Tsp of sugar in vat at time $t$

$S(0) = 0$

$$\frac{dS}{dt} = \frac{6 \text{ Tsp}}{\text{gal}} \cdot \frac{3 \text{ gal}}{\text{min}} + \frac{7 \text{ Tsp}}{\text{gal}} \cdot \frac{2 \text{ gal}}{\text{min}} - \frac{S(t)}{100} \cdot \frac{5 \text{ gal}}{\text{min}}$$

IVP
$$\begin{cases} \dfrac{dS}{dt} = 18 + 14 - \dfrac{S}{20} = \dfrac{640 - S}{20} \\ S(0) = 0 \end{cases}$$

1 pt

3 pts

b.) (7 pts) : Solve the initial value problem

$$\int \frac{dS}{640-S} = \int \frac{1}{20}\, dt$$

$$-\ln|640-S| = \frac{1}{20}t + c$$

$$\ln|640-S| = -\frac{1}{20}t + c$$

$$640 - S = ce^{-\frac{1}{20}t}$$

$$\boxed{S(t) = 640 + ce^{-\frac{1}{20}t}}$$ 5pts

$$S(0) = 0 = 640 + C \Rightarrow \boxed{C = -640}$$ 2pts

$$\boxed{S(t) = 640\left(1 - e^{-\frac{1}{20}t}\right)}$$

$$S(30) \approx 640\,(.77687) = 497.2 \text{ TSP}$$

c.) (2 pts): How much sugar will be in the tank after 30 minutes?

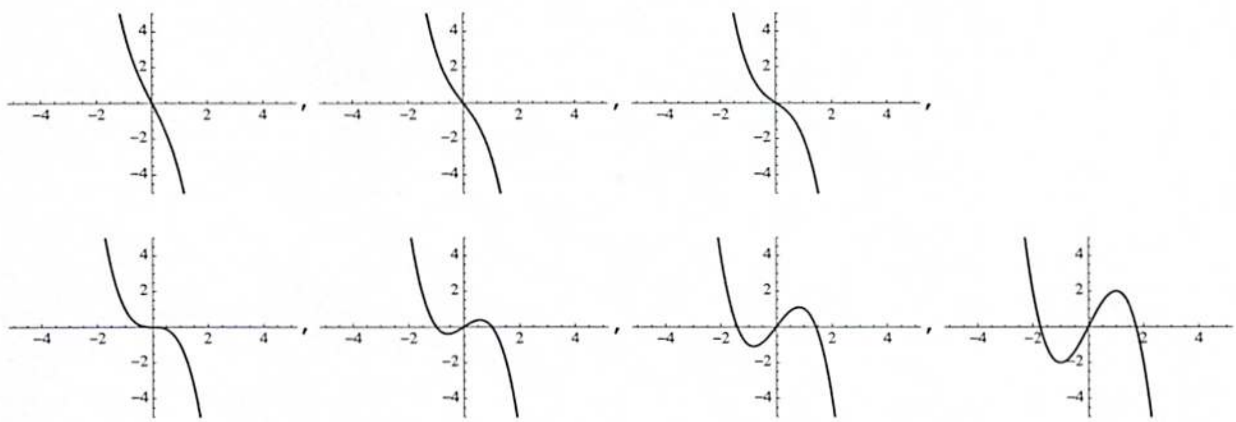
$S(30) \approx 497.2$ Tsps

d.) (2 pts) : What is $\lim_{t\to\infty} S(t)$ $= 640$ TSP

5. (10 pts) Given the one parameter family of differential equations:

$\frac{dy}{dt} = Ay - y^3$ and a parameter study for values of the parameter A ranging from A = -3 to 3 create a bifurcation diagram and indicate the bifurcation value(s).

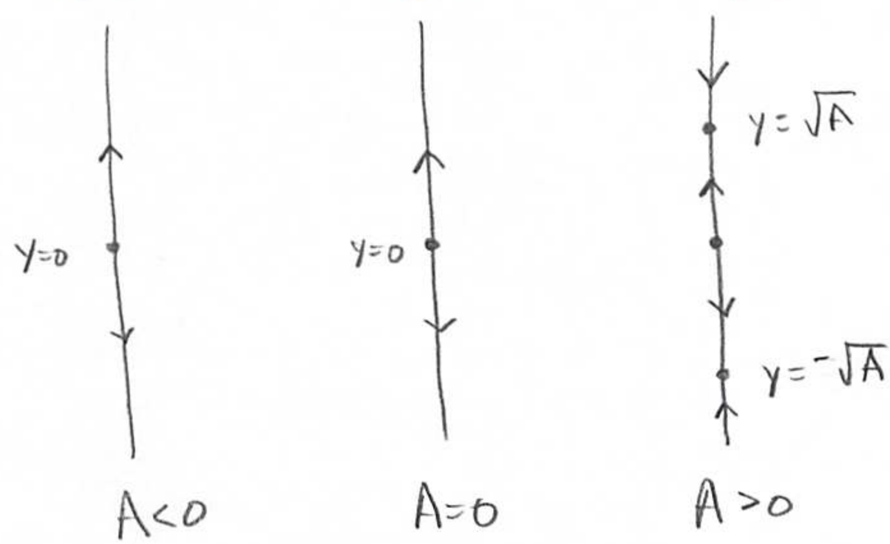
A = -3 A = -2 A=-1



A = 0 A = 1 A = 2 A = 3

$$\frac{dy}{dt} = y(A - y^2) \qquad \frac{dy}{dt} = 0 \Rightarrow y = 0 \text{ or } y = \pm\sqrt{A}$$

3pts 2pts 3pts




$\boxed{A=0 \text{ is bifurcation value}}$ 2pts

6. (8 pts) Use Euler's Method with a step size of 0.25 to approximate the solution of the initial value problem $\frac{dy}{dt} = y^2 - 2t$ and $y(0) = 1$ over the interval $0 \le t \le .75$ .

Create a table that shows how you computed the approximate y values for values for t= .25, .5, and .75

Use 6 places of accuracy

| $K$ | $t_K$ | $Y_K$ | $f(t_K, Y_K) = Y_K^2 - 2t_K$ | $Y_{K+1} = Y_K + f(t_K, Y_K)(.25)$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 1 pt | $1 + (1)(.25) = 1.25$ 1 pt |
| 1 | .25 | 1.25 1 pt | $(1.25)^2 - 2(.25) = 1.0625$ 1 pt | $1.25 + (1.0625)(.25) = 1.515625$ 1 pt |
| 2 | .50 | 1.515625 1 pt | 1.297119 1 pt | 1.839905 1 pt |
| 3 | .75 | 1.839905 | | |