Name: Prof D

Discussion Section : ______

MA 226 Section B – Exam 1

Question Number	Possible Points	Student Score
1	12	
2a	8	
2b	8	
2c	8	
2d	8	
3	16	
4	16	
5	16	
6	8	
Total Points	100	

You must show your work to receive full credit

Discussion Sections:

B2: Tuesday 4:30-5:30

B3: Tues : 3:30-4:30

B4: Weds: 9-10

B5: Weds: 10-11

B6: Weds: 4:30-5:30

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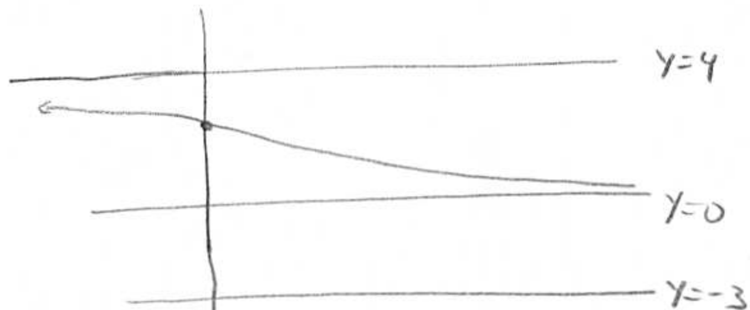
1. Short Answer (12 pts)

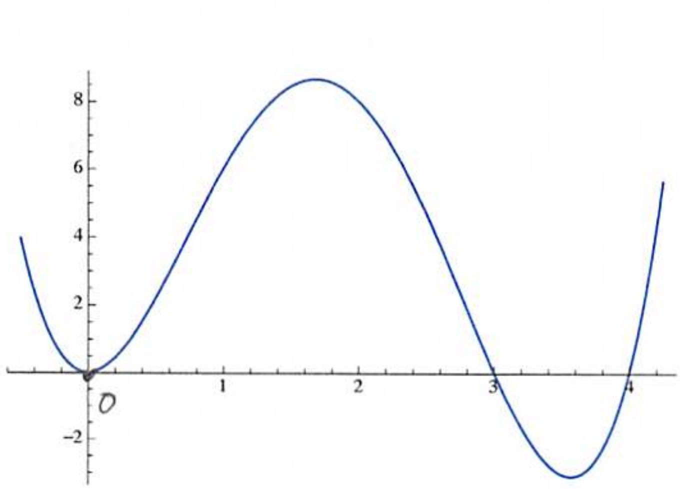
a) Given the differential equation : $\frac{dy}{dt} = 2y(y+3)(y-4)$. Let $y_1(t)$ be a solution that satisfies the initial condition $y_1(0) = 2$. Evaluate

$\lim_{t \to \infty} y_1(t) =$ 0 $\qquad$ $\lim_{t \to -\infty} y_1(t) =$ 4



b) Find the equilibrium solutions of the differential equation
$\frac{dy}{dt} = \frac{(t^2-4)(y^2-1)}{y^2-9}$ $\qquad$ y = 1 and y = -1

c) Draw the phase line that corresponds to the differential equation $\frac{dy}{dt} = f(y)$ where $f(y)$ is given by the graph below



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2. Solving differential equations and initial value problems

a) (8 pts) Find the general solution of the equation: $\frac{dy}{dt} = 2y(2-y)$

$$\int \frac{1}{y(2-y)}\,dy = \int 2\,dt$$

$$\frac{1}{2}\int \frac{1}{y} + \frac{1}{2-y}\,dy = \int 2\,dt$$

$$\ln|y| - \ln|2-y| = 4t + C$$

$$\ln\left|\frac{y}{2-y}\right| = 4t + C$$

$$\left|\frac{y}{2-y}\right| = c\cdot e^{4t} \qquad c>0$$

$$\frac{y}{2-y} = ce^{4t} \qquad -\infty < C < \infty$$

$$y = 2ce^{4t} - yce^{4t}$$

$$y(1+ce^{4t}) = 2ce^{4t}$$

$$\boxed{y(t) = \frac{2ce^{4t}}{1+ce^{4t}}}$$

genrl soln

$$\boxed{\begin{array}{l} y(t)=0 \\ y(t)=2 \end{array}}$$

$$\frac{1}{y(2-y)} = \frac{A}{y} + \frac{B}{2-y}$$

$$= \frac{A(2-y) + By}{y(2-y)} = \frac{2A + y(B-A)}{y(2-y)}$$

$$A = \frac{1}{2} \quad B = \frac{1}{2}$$

$$\frac{1}{2y} + \frac{1}{2(2-y)} = \frac{(2-y)+y}{2y(2-y)} = \frac{\cancel{2}\,1}{\cancel{2}y(2-y)} \checkmark$$

Check

$$\frac{dy}{dt} = \frac{8ce^{4t}(1+ce^{4t}) - 2ce^{4t}(4ce^{4t})}{(1+ce^{4t})^2}$$

$$= \frac{8ce^{4t} + \cancel{8c^2e^{8t}} - \cancel{8c^2e^{8t}}}{(1+ce^{4t})^2}$$

$$= \frac{8ce^{4t}}{(1+ce^{4t})^2}$$

$$2y(2-y) = \left(\frac{4ce^{4t}}{1+ce^{4t}}\right)\left(2 - \frac{2ce^{4t}}{1+ce^{4t}}\right)$$

$$= \frac{4ce^{4t}}{1+ce^{4t}}\left(\frac{2+\cancel{2ce^{4t}}-\cancel{2ce^{4t}}}{(1+ce^{4t})}\right)$$

$$= \frac{4ce^{4t}\cdot 2}{(1+ce^{4t})^2} \checkmark$$

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b) (8 pts) Solve the initial value problem $\frac{dy}{dt} + 2y = \cos(4t)$ with $y(0) = 0$

$y(t) = y_h(t) + y_p(t)$

$\frac{dy_h}{dt} + 2y_h = -2ke^{-2t} + 2ke^{-2t} = 0$ ✓

$y_h(t) = ke^{-2t}$

$y_p(t) = A\cos(4t) + B\sin(4t)$

$\frac{dy_p}{dt} + 2y_p = -4A\sin 4t + 4B\cos 4t + 2A\cos 4t + 2B\sin 4t = \cos(4t)$

$(-4A + 2B)\sin(4t) + (4B + 2A)\cos 4t = \cos(4t)$

$2A + 4B = 1$
$-4A + 2B = 0$ $\Rightarrow$ $4A + 8B = 2$
$-4A + 2B = 0$

$4A = 2B$ $\qquad$ $10B = 2$

$A = \frac{1}{2}B$ $\qquad$ $\boxed{B = \frac{2}{10} \quad A = \frac{1}{10}}$

check:

$\frac{dy_p}{dt} + 2y_p = -4\frac{1}{10}\sin 4t + 4\cdot\frac{2}{10}\cos 4t + 2\frac{1}{10}\cos 4t + 2\cdot\frac{2}{10}\sin 4t$

$= \frac{10}{10}\cos 4t = \cos 4t$ ✓

$y(t) = ke^{-2t} + \frac{1}{10}\cos 4t + \frac{2}{10}\sin 4t$

$y(0) = k + \frac{1}{10} = 0 \quad \Rightarrow \quad k = -\frac{1}{10}$

$$\boxed{y(t) = -\frac{1}{10}e^{-2t} + \frac{1}{10}\cos 4t + \frac{2}{10}\sin 4t}$$

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c) (8 pts) Solve the initial value problem $\frac{dy}{dt} - \frac{2}{t} y = t^2 e^{3t}$ with $y(1) = 0$

integrating factor $\mu = e^{\int -\frac{2}{t} dt} = e^{-2\ln|t|} = e^{\ln \frac{1}{t^2}} = \frac{1}{t^2}$

$$\frac{1}{t^2} \frac{dy}{dt} - \frac{2}{t^3} y = e^{3t}$$

$$\frac{d}{dt}\left(\frac{1}{t^2} \cdot y \right) = e^{3t}$$

$$\int \frac{d}{dt}\left(\frac{1}{t^2} \cdot y \right) dt = \int e^{3t} dt$$

$$\frac{1}{t^2} \cdot y = \frac{1}{3} e^{3t} + c$$

$$\boxed{y(t) = \frac{1}{3} t^2 e^{3t} + ct^2}$$

check: $\frac{dy}{dt} = \frac{2}{3} t e^{3t} + t^2 e^{3t} + 2ct$

$$-\frac{2}{t} y = -\frac{2}{3} t e^{3t} - 2ct$$

$$\frac{dy}{dt} - \frac{2}{t} y = t^2 e^{3t} \quad \checkmark$$

$$y(1) = 0 = \frac{1}{3} e^3 + c$$

$$c = -\frac{1}{3} e^3$$

$$y(t) = \frac{1}{3} t^2 e^{3t} - \frac{1}{3} e^3 t^2 = \boxed{\frac{1}{3} t^2 \left(e^{3t} - e^3 \right)}$$

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d) (8 pts) Solve the initial value problem $\frac{dy}{dt} = 3y + 2e^{3t}$ with $y(0) = 4$

$y(t) = y_h(t) + y_p(t)$

$y_h(t) = ke^{3t}$

$y_p(t) = Ate^{3t}$

$\frac{dy_p}{dt} = Ae^{3t} + 3Ate^{3t}$

$\frac{dy_p}{dt} = 3y_p + 2e^{3t}$

$Ae^{3t} + \cancel{3Ate^{3t}} = \cancel{3Ate^{3t}} + 2e^{3t}$

$A = 2$

$\boxed{y_p(t) = 2te^{3t}}$

$\boxed{y(t) = ke^{3t} + 2te^{3t}}$

$y(0) = 4 = k + 0 \Rightarrow k = 4$

$\boxed{y(t) = 4e^{3t} + 2te^{3t}}$

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3. (16 pts) Matching slope fields

(i) $\frac{dy}{dt} = y^2 + y$ (ii) $\frac{dy}{dt} = y^2 - y$ (iii) $\frac{dy}{dt} = y^3 + y^2$ (iv) $\frac{dy}{dt} = 2 - t^2$

(v) $\frac{dy}{dt} = ty + ty^2$ (vi) $\frac{dy}{dt} = t^2 + t^2 y$ (vii) $\frac{dy}{dt} = t + ty$ (viii) $\frac{dy}{dt} = t^2 - 2$

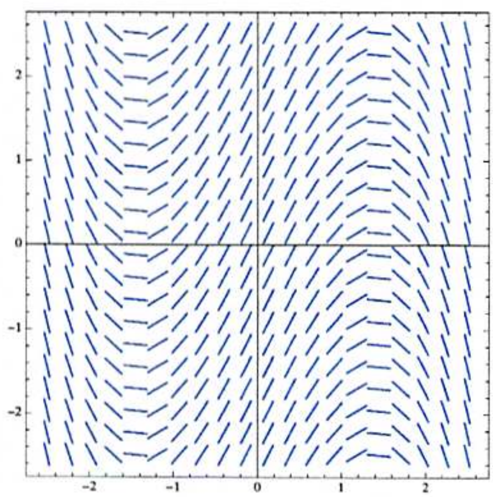
$ty(1+y)$ $t^2(1+y)$ $t(1+y)$

Slope Field A equation: (iv)

Reason:

independent of y

t=0 ⇒ positive slopes



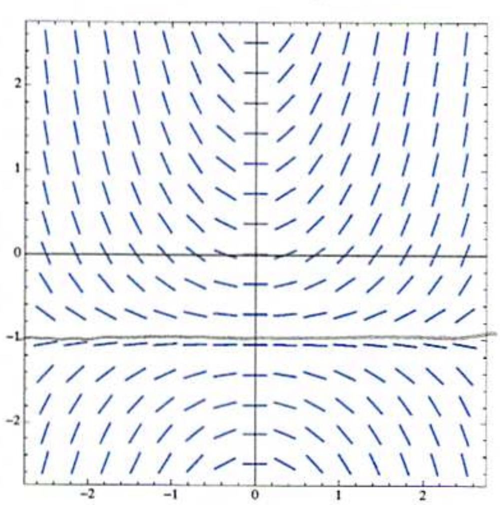
Slope Field B equation: vii

Reason

y=-1 only equilibrium

quadrant II t<0 , y>0

slopes are negative



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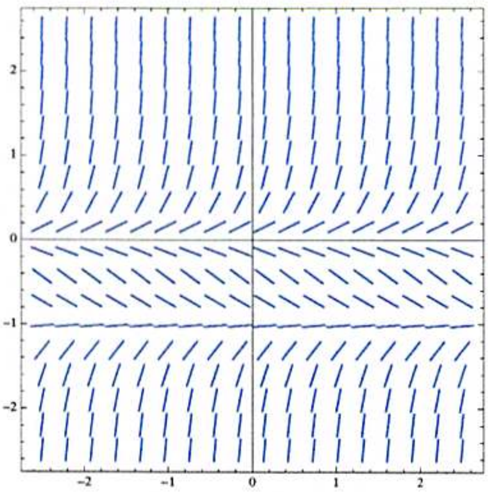
$y(y+1)$ $\qquad$ $y(y-1)$ $\qquad$ $y^2(y+1)$

(i) $\frac{dy}{dt} = y^2 + y$ (ii) $\frac{dy}{dt} = y^2 - y$ (iii) $\frac{dy}{dt} = y^3 + y^2$ (iv) $\frac{dy}{dt} = 2 - t^2$

(v) $\frac{dy}{dt} = ty + ty^2$ (vi) $\frac{dy}{dt} = t^2 + t^2 y$ (vii) $\frac{dy}{dt} = t + ty$ (viii) $\frac{dy}{dt} = t^2 - 2$

$ty(1+y)$ $\qquad$ $t^2(1+y)$

Slope Field C equation: __i__ Reason

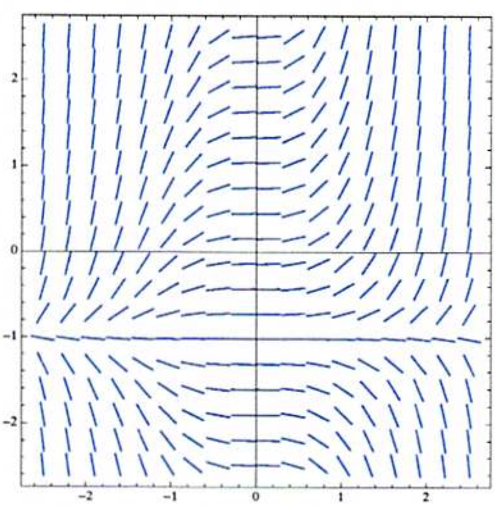


equilibrium $y = 0, y = -1$

$y < -1$ slopes are positive.

$0 < y < -1$ slopes are negative

Slope Field D equation: __vi__ Reason



$y = -1$ equilibrium pt

quad. II $t < 0, y > 0$

slopes are positive

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4. (16 pts) A college professor contributes $8,000 per year into her retirement fund by making many small deposits throughout the year. The fund grows at a rate of 7% per year compounded continuously. After 30 years, she retires and begins withdrawing from her fund at a rate of $5,000 per month. If she does not make any deposits after retirement, how long will the money last? [Hint: Solve the problem in two steps, before retirement and after]

$$\frac{dy}{dt} = .07y + 8000 \qquad y(0) = 0$$

$$y_h(t) = Ke^{.07t} \qquad y_p(t) = \frac{-8000}{.07} = 114285$$

$$y(t) = Ke^{.07t} - 114285 \qquad y(0) = 0 \Rightarrow K = 114285$$

$$\boxed{y(t) = 114285\left(e^{.07t} - 1\right)} \qquad y(30) = 818985$$

$$\frac{dy}{dt} = .07y - (5000)12 \qquad y(0) = 818985$$

$$y_p(t) = \frac{60000}{.07} = 857142$$

$$y(t) = Ke^{.07t} + 857142 \qquad y(0) = 818985$$

$$y(0) = K + 857142 = 818985$$

$$K = -38157$$

$$\boxed{y(t) = -38157e^{.07t} + 857142}$$

Ask when $y(t) = 0 \Rightarrow$

$$38157e^{.07t} = 857142$$

$$e^{.07t} = 22.46$$

$$.07t = \ln(22.46)$$

$$t = \frac{\ln(22.46)}{.07} = 44.45 \text{ years}$$

$$\boxed{44.45 \text{ years}}$$

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5. (16 pts) A cup of hot chocolate is initially $170^{\circ}F$ and is left in a room with an ambient temperature of $65^{\circ}F$. Suppose at time $t = 0$ it is cooling at a rate of 15° per minute.

a.) Assume the Newton's law of cooling applies: The rate of cooling is proportional to the difference between the current temperature and the ambient temperature. Write an initial value problem that models the temperature of the hot chocolate.

$$\frac{dT}{dt} = K(T - T_A) \qquad T(0) = 170^{\circ}F \qquad T_A = 65^{\circ}F$$

$$\left.\frac{dT}{dt}\right|_{t=0} = -15 = K(170-65) \qquad K = -\frac{15}{105} = -\frac{1}{7}$$

b.) Find a solution of the initial value problem.

$$\int \frac{1}{T-65}\,dT = \int -\frac{1}{7}\,dt$$

$$\ln|T-65| = -\frac{1}{7}t + c$$

$$|T-65| = e^{-\frac{1}{7}t} \cdot e^{c}$$

$$T - 65 = ce^{-\frac{1}{7}t}$$

$$T(t) = 65 + ce^{-\frac{1}{7}t}$$

$$T(0) = 170 = 65 + ce^{-\frac{1}{7}t}$$

$$105 = c$$

$$\boxed{T(t) = 65 + 105e^{-\frac{1}{7}t}}$$

c.) How long does it take for the hot chocolate to cool to $100^{\circ}F$? (show your work)

As $T(t) = 100F^{\circ}$

$$100 = 65 + 105e^{-\frac{1}{7}t}$$

$$\frac{35}{105} = e^{-\frac{1}{7}t}$$

$$-\frac{1}{7}t = \ln\left(\frac{35}{105}\right)$$

$$t = -7\ln\left(\frac{35}{105}\right)$$

$$\boxed{t = 7.69 \text{ minutes}}$$

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6. (8 pts) Use Euler's Method with a step size of 0.5 to approximate the solution of the initial value problem $\frac{dy}{dt} = 2t - y$ and y(0)= 2 over the interval $0 \le t \le 2$.

Create a table that shows how you computed the values for t=.5, 1, 1.5, and 2

K	t_K	y_K	$f(t_K, y_K)$	$y_{K+1} = y_K + (.5) f(t_K, y_K)$
0	0	2	-2	$2 + (.5)(-2) = 1$
1	.5	1	0	$1 + .5(0) = 1$
2	1	1	1	$1 + .5(1) = 1.5$
3	1.5	1.5	1.5	$1.5 + .5(1.5) = 2.25$
4	2	2.25		