Sec 1.8 Linear Equations

A first-order differential equation is linear if it can be written in the form

$$\frac{dy}{dt} = a(t)\cdot y + b(t)$$

where $a(t)$ and $b(t)$ are arbitrary functions of t.

Examples:

$$\frac{dy}{dt} = t^2 y + \cos t$$

$$\frac{dy}{dt} = \frac{e^{4\sin t}}{t^3+7t}\cdot y + 23t^3 - 7t^2 + 3$$

$$\frac{dy}{dt} = 2y + 8$$

The term linear refers to the fact that the dependent variable y appears in the equation only to the first power.

Examples of nonlinear 1st order diff. eqns

$$\frac{dy}{dt} = y^2$$

$$\frac{dy}{dt} = t\cdot \sin(y)$$

Additional Terminology

$$\frac{dy}{dt} = a(t)\cdot y + b(t)$$

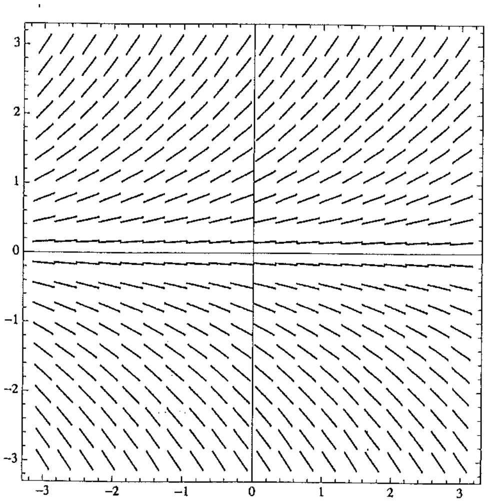
- If $b(t)=0$ for all t the equation is called homogeneous or unforced. Otherwise it is called nonhomogeneous or forced.
- If $a(t)$ is a constant the linear differential equation is called a constant-coefficient equation $\frac{dy}{dt} = \lambda y + b(t)$ λ = constant

The Linearity Principle

If $y_h(t)$ is a solution of the homogeneous linear equation $\frac{dy}{dt} = a(t)\cdot y$ then any constant multiple of $y_h(t)$ is also a solution. That is $k\cdot y_h(t)$ is a solution for any constant k.

Note: $y_h(t)$ is not an equilibrium solution

Verify Linearity Principle



Note: The Linearity Principle only applies to homogeneous linear equations.

Consider $\frac{dy}{dt} = y^2$ with solution

$y_1(t) = \frac{1}{1-t}$

Extended Linearity Principle

Consider the nonhomogeneous equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y$$

1.) If $y_h(t)$ is a solution of the homogeneous equation and $y_p(t)$ is any solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

Extended linearity Principle (continued)

2.) Suppose that $y_p(t)$ and $y_q(t)$ are both solutions to the nonhomogeneous equation. Then $y_q(t) - y_p(t)$ is a solution of the associated homogeneous solution.

Therefore, if $y_h(t)$ is nonzero, $k y_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.

Fact: Given a first order linear nonhomogeneous differential equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

the general solution has the form

$$\underbrace{k y_h(t)}_{\text{general solution of the associated homogeneous equation}} + \underbrace{y_p(t)}_{\text{any particular solution of the nonhomogeneous equation.}}$$

Solving Linear Constant Coefficient Nonhomogeneous Equations

We are going to focus on equations of the form

$$\frac{dy}{dt} = ky + b(t) \qquad k \text{ is const.}$$

$b(t)$ is called the nonhomogeneous term or the forcing term.

We will use a method called undetermined coefficients to find a particular solution when the forcing term is one of the following:

- $b(t) = \beta e^{\delta t}$ where β and δ are constants
- $b(t) = \beta \sin(\delta t)$ or $\beta \cos(\delta t)$
- $b(t) = \alpha t + \beta$

Case 1 $b(t) = \beta e^{\delta t}$

Example: $\frac{dy}{dt} = -2y + e^{3t}$

Case 2: $b(t) = \beta \cos(\delta t)$ or $\beta \sin(\delta t)$

Example: $\frac{dy}{dt} = -2y + \cos(3t)$

Case 3: $b(t) = \alpha t + \beta$

Example: $\frac{dy}{dt} = -2y + t$

Case 1 (Revisited) $b(t) = \beta e^{\delta t}$

Caution!

Example: $\frac{dy}{dt} = -2y + 3e^{-2t}$