

Sec 1.7 Bifurcations

A differential equation that depends on a parameter bifurcates if there is a qualitative change in the behavior of solutions as the parameter changes.

Example: $\frac{dy}{dt} = y^2 - 2y + \mu$

$\underbrace{\hspace{10em}}_{f_\mu(y)}$

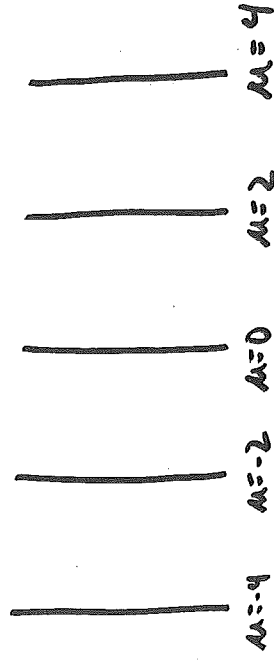
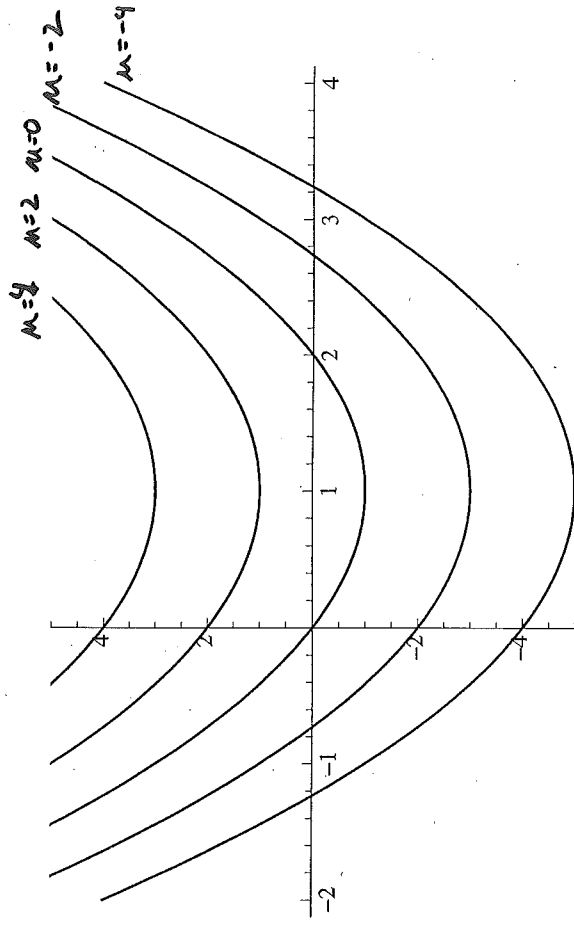
To emphasize the dependence of the right hand side on the parameter μ we make it a subscript.

$$\frac{dy}{dt} = f_\mu(y)$$

This notation refers to a collection of differential equations (one for each value of μ) and is called a one parameter family of differential equations.

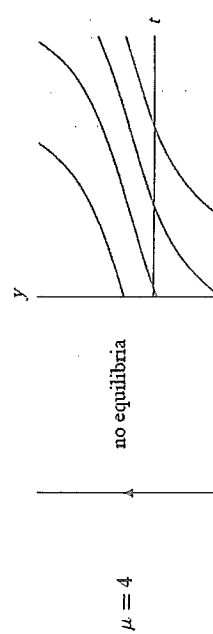
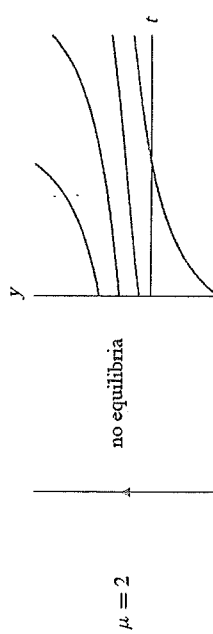
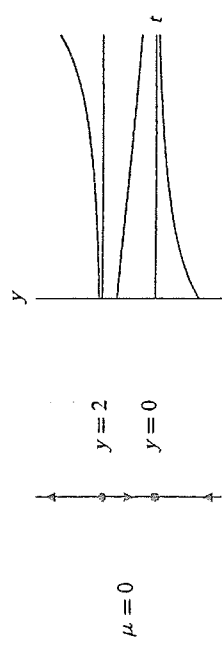
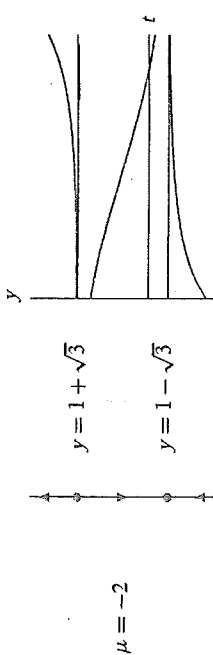
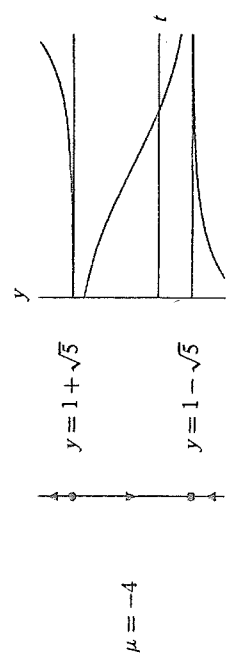
Example 1

Consider the graph of $f_\mu(y) = y^2 - 2y + \mu$ for $\mu = -4, -2, 0, 2, 4$



Consider the roots of $f_\mu(y)$ in terms of the parameter μ .

Ask $f_\mu(y) = 0 \Rightarrow y^2 - 2y + \mu = 0$



Example 2

$$\frac{dy}{dt} = y^3 - \alpha y = g_\alpha(y)$$

$$g_\alpha(y) = y^3 - \alpha y = y(y^2 - \alpha)$$

Three cases:

• $\alpha = 0$ $g_0(y) = y^3 \Rightarrow y=0$ is equilibrium point

• $\alpha < 0$ $g_\alpha(y) = y(y^2 - \alpha)$ $y=0$ is only equilibrium point

positive if $y > 0$

$g_\alpha(y) > 0$ if $y > 0$

$g_\alpha(y) < 0$ if $y < 0$

• $\alpha > 0$ $g_\alpha(y) = y(y^2 - \alpha) = 0$ when $y=0$ or $y = \pm\sqrt{\alpha}$

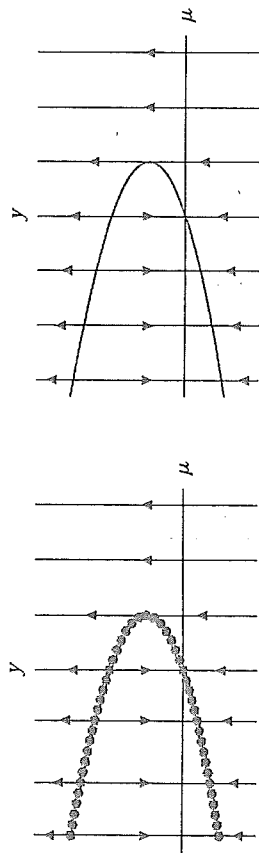
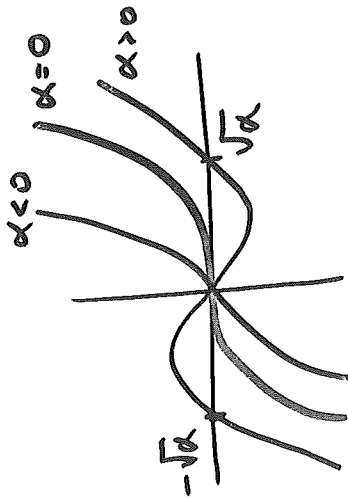
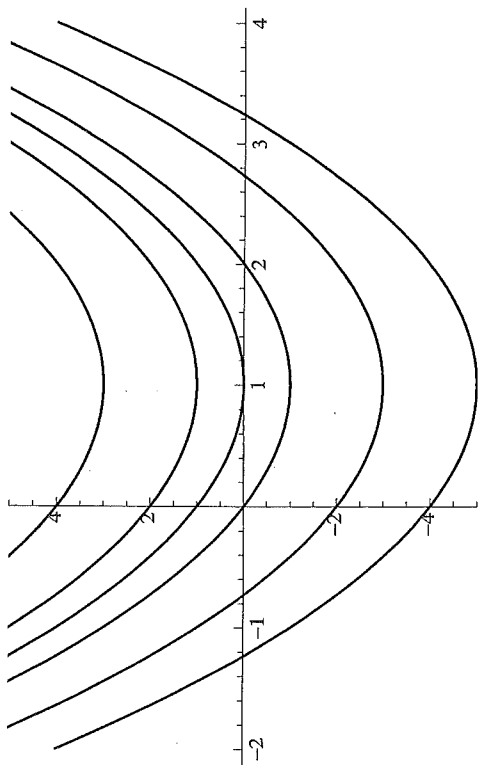


Figure 1.83

Bifurcation diagram for the differential equation $dy/dt = f_\mu(y) = y^2 - 2y + \mu$. The horizontal axis is the μ -value and the vertical lines are the phase lines for the differential equations with the corresponding μ -values.



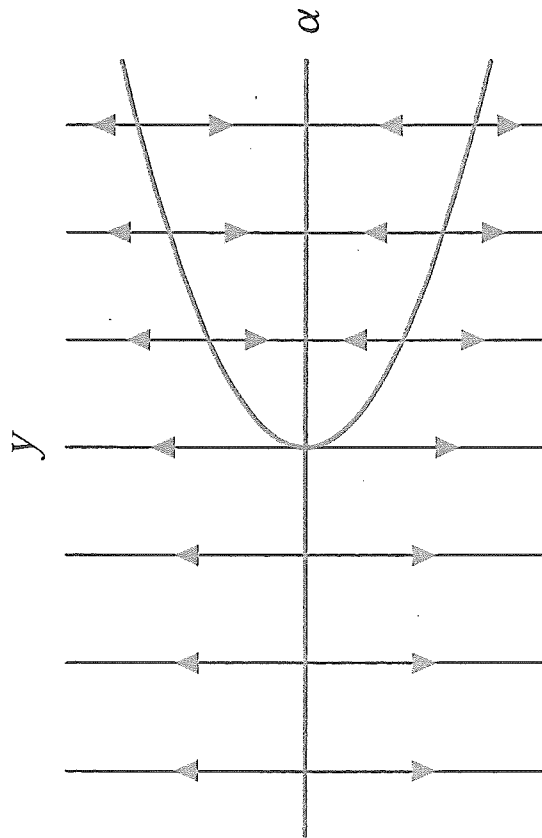
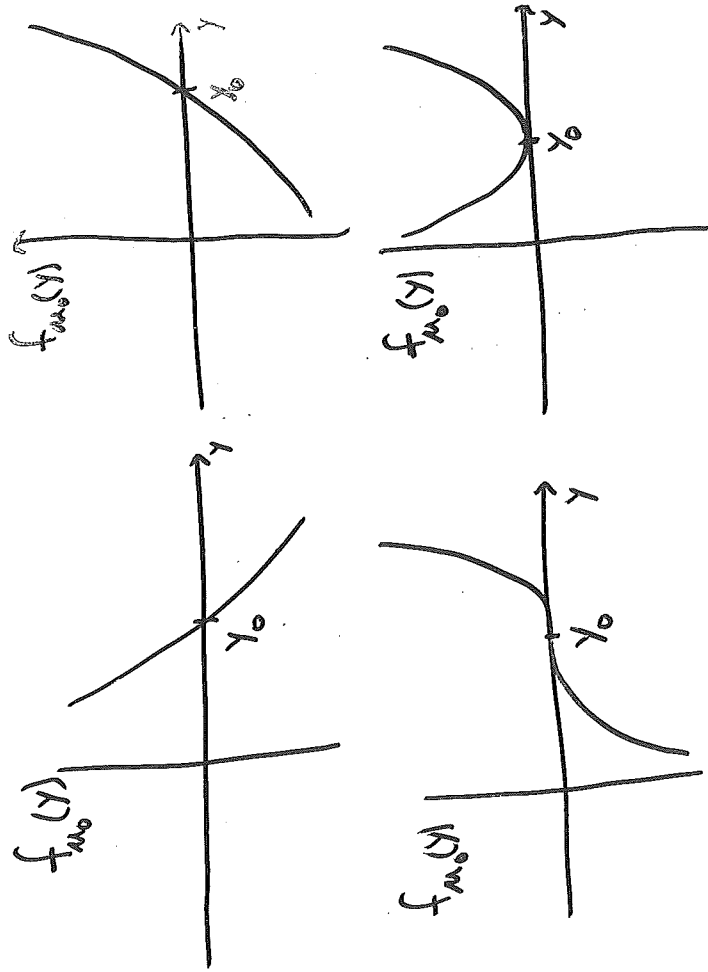
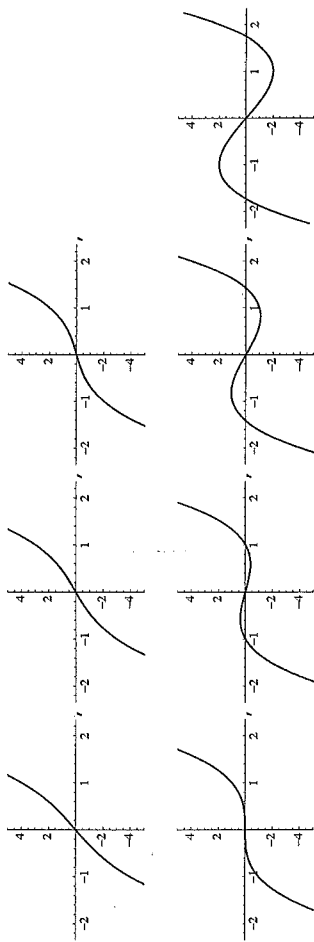
$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

How to find bifurcation values

If $\frac{dy}{dt} = f_{\mu}(y)$

We are looking for situations where

$f_{\mu_0}(y_0) = 0$ and $f'_{\mu_0}(y_0) = 0$



Example 3

$$\frac{dy}{dt} = y(1-y)^2 + \mu = f_{\mu}(y)$$

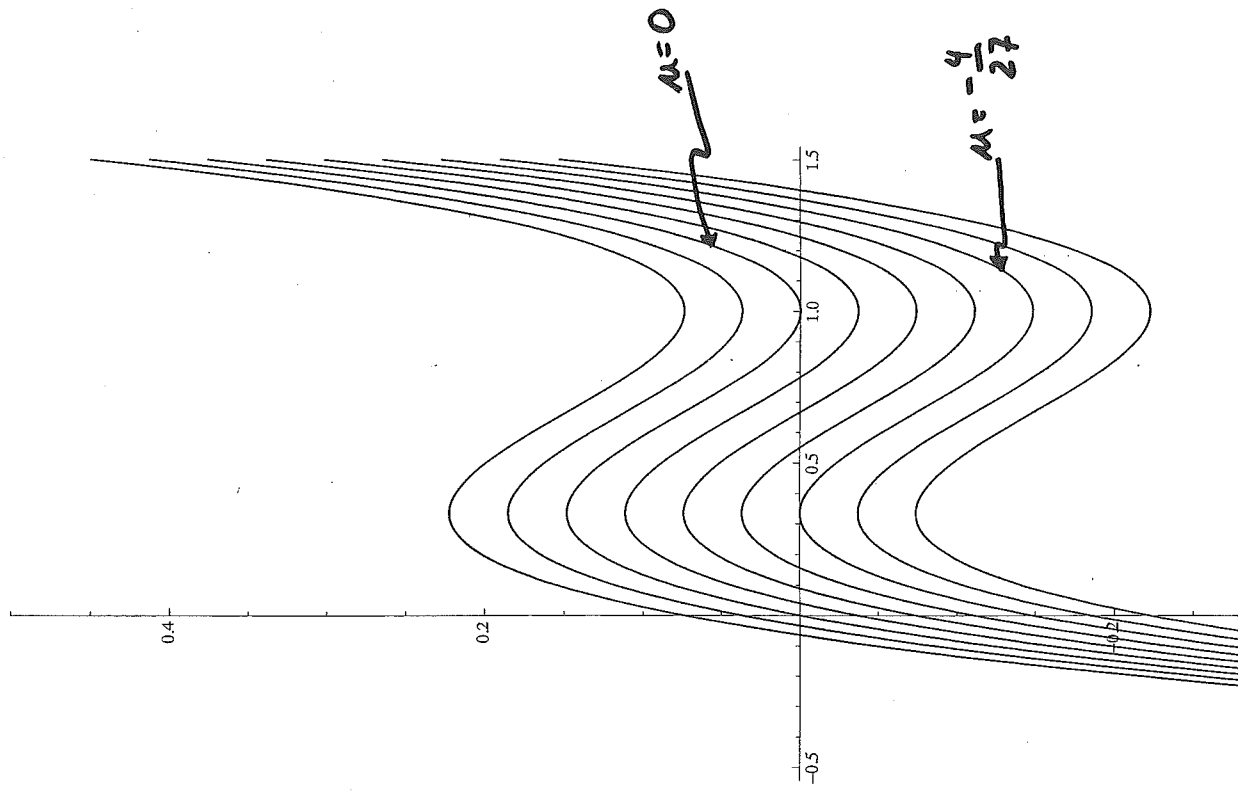
Looking for y and μ values so that

$$f_{\mu_0}(y_0) = 0 \quad \text{and} \quad f'_{\mu_0}(y_0) = 0$$

- First find y values where $f'_{\mu}(y) = 0$

$$f'_{\mu}(y) = 1(1-y)^2 - 2y(1-y) + 0$$

- Find μ so that $f_{\mu}(y_0) = 0$



Application - Sustainability

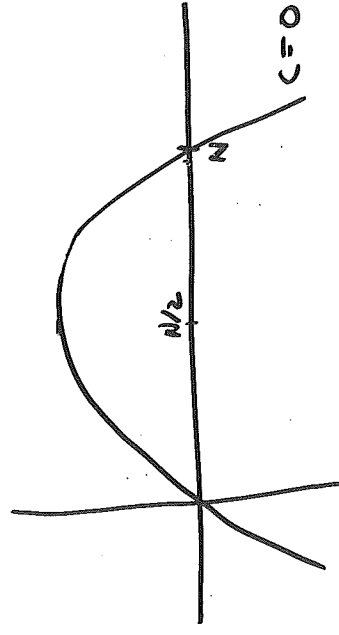
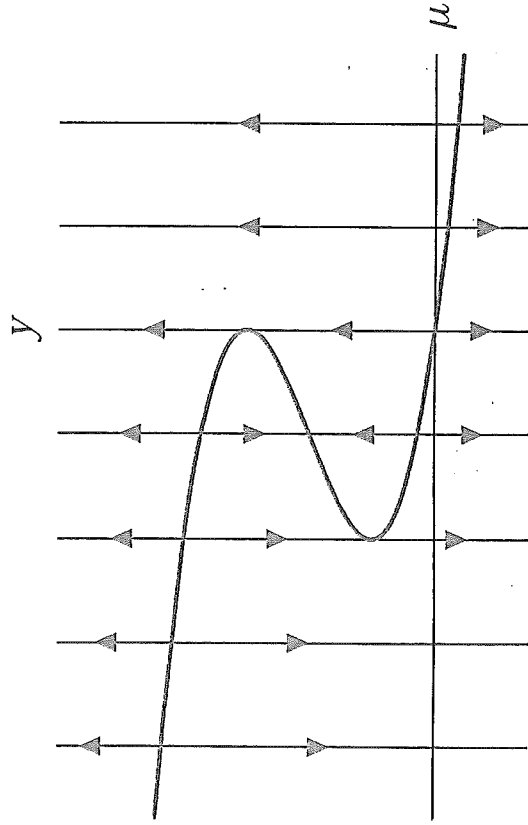
Suppose that we model a population of fish with the logistic model

$$\frac{dP}{dt} = KP \left(1 - \frac{P}{N}\right)$$

Suppose that fishing removes a constant number of fish each year which we call C .

The modified model for fish population is

$$\begin{aligned}\frac{dP}{dt} &= KP \left(1 - \frac{P}{N}\right) - C \\ &= -\frac{K}{N}P^2 + KP - C\end{aligned}$$



How to find the bifurcation value C

for $\frac{dP}{dt} = kP(1 - \frac{P}{N}) - C = f_c(P)$

Ask: $f'_c(P) = 0$ and $f_c(P) = 0$

$$f_c(P) = -\frac{k}{N}P^2 + kP - C$$

$$f'_c(P) = -\frac{2k}{N}P + k$$

$$f'_c(P) = 0$$

