Sec 1.7 Bifurcations

A differential equation that depends on a parameter **bifurcates** if there is a qualitative change in the behavior of solutions as the parameter changes.

Example:

$$\frac{dy}{dt} = \underbrace{y^2 - 2y + \mu}_{f_\mu(y)}$$

To emphasis the dependence of the right hand side on the parameter μ we make it a subscript.

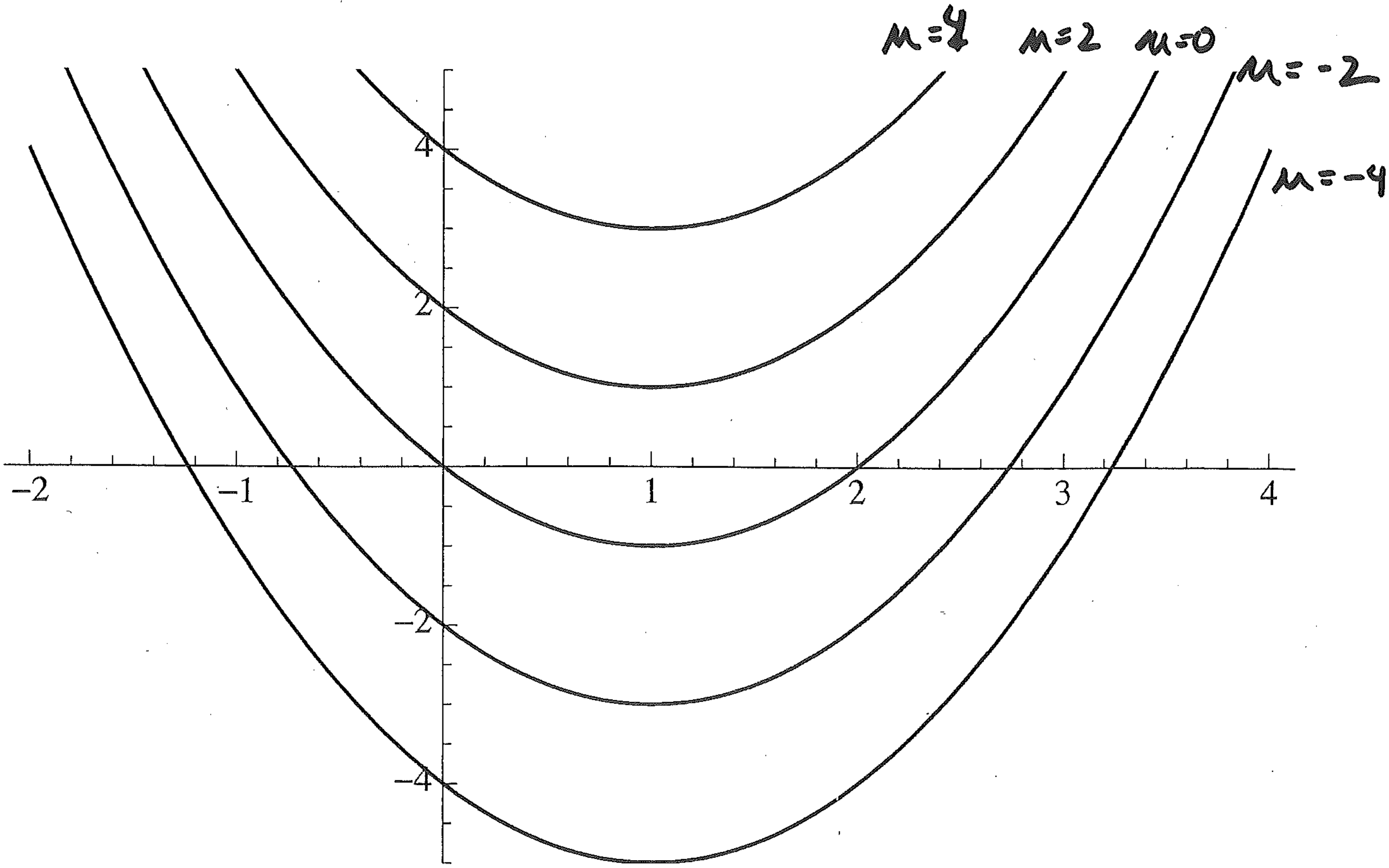
$$\frac{dy}{dt} = f_\mu(y)$$

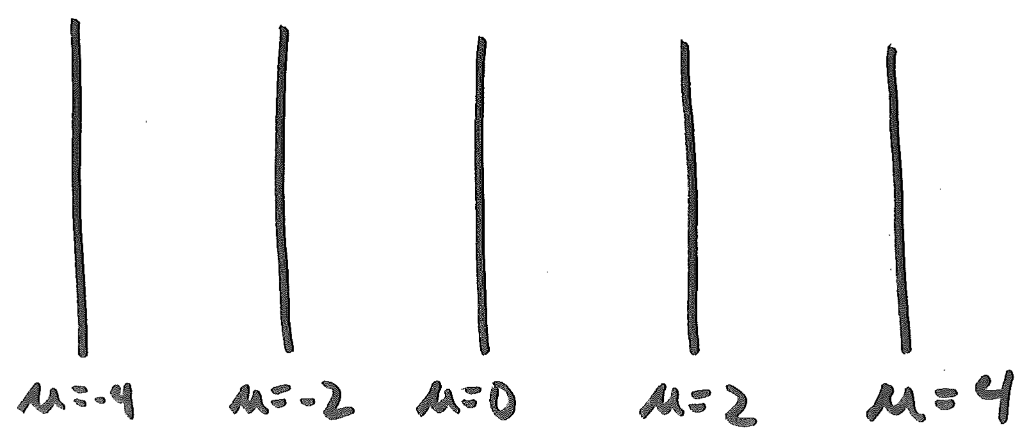
This notation refers to a collection of differential equations (one for each value of μ) and is called a **one parameter family** of differential equations.

Example 1

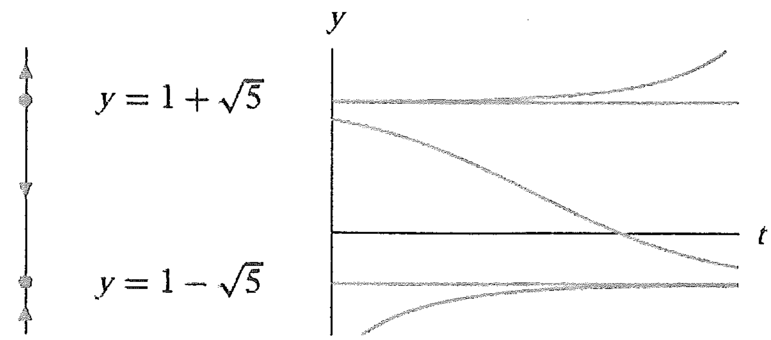
Consider the graph of $f_\mu(y) = y^2 - 2y + \mu$

for $\mu = -4, -2, 0, 2, 4$

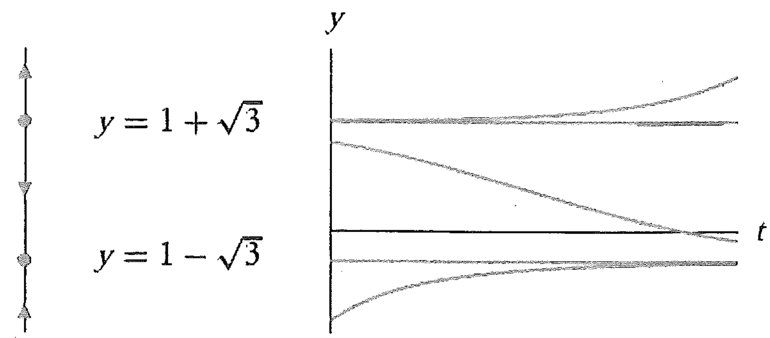




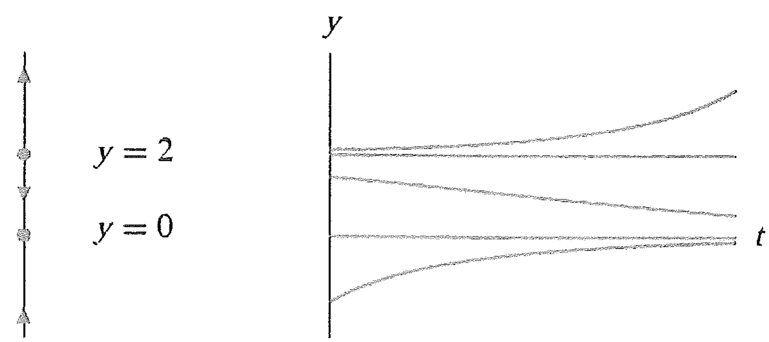
$\mu = -4$



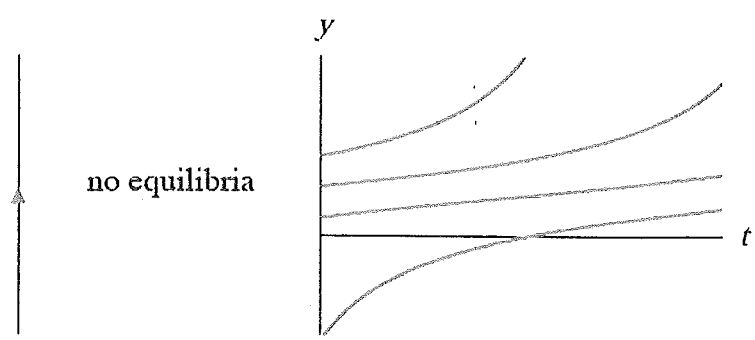
$\mu = -2$



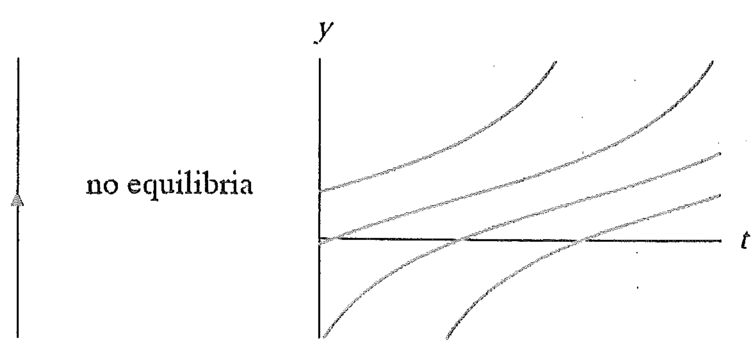
$\mu = 0$



$\mu = 2$

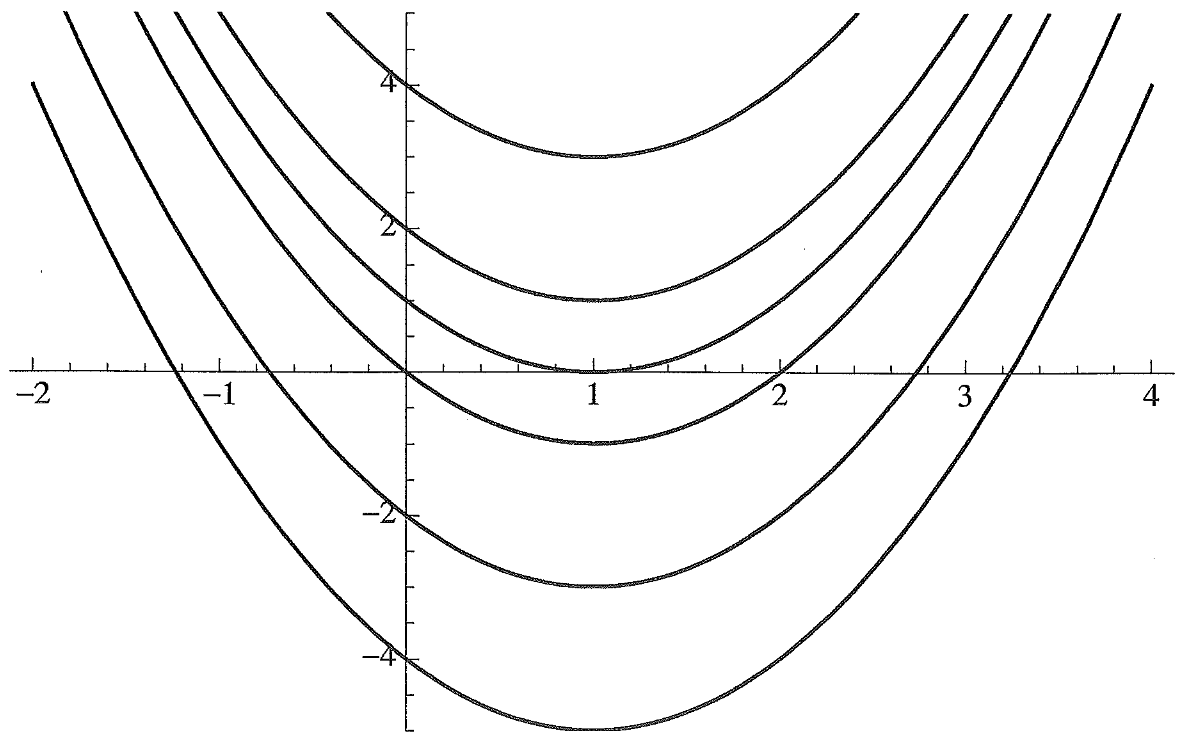


$\mu = 4$



Consider the roots of $f_\mu(y)$ in terms of the parameter μ.

Ask $f_\mu(y) = 0 \Rightarrow y^2 - 2y + \mu = 0$



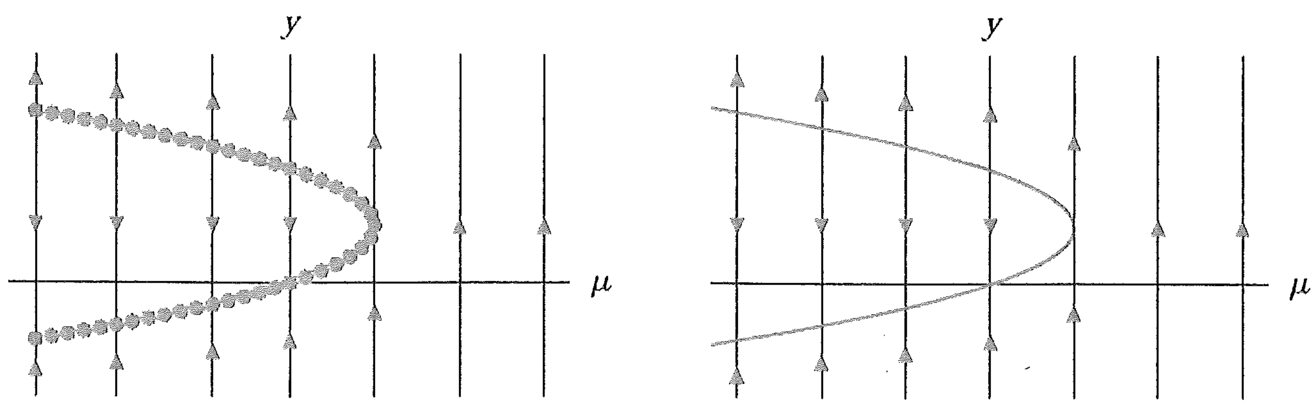


Figure 1.83
Bifurcation diagram for the differential equation $dy/dt = f_\mu(y) = y^2 - 2y + \mu$. The horizontal axis is the μ-value and the vertical lines are the phase lines for the differential equations with the corresponding μ-values.

Example 2

$$\frac{dy}{dt} = y^3 - \alpha y = g_\alpha(y)$$

$$g_\alpha(y) = y^3 - \alpha y = y(y^2 - \alpha)$$

Three cases:

- $\alpha = 0$ $\quad g_0(y) = y^3 \quad \Rightarrow y = 0$ is equilibrium point

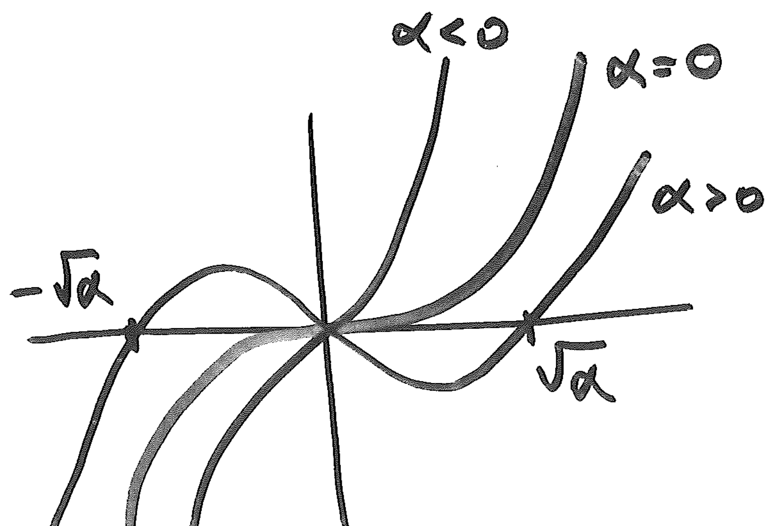
- $\alpha < 0$ $\quad g_\alpha(y) = y\underbrace{(y^2 - \alpha)}_{\text{positive}}$

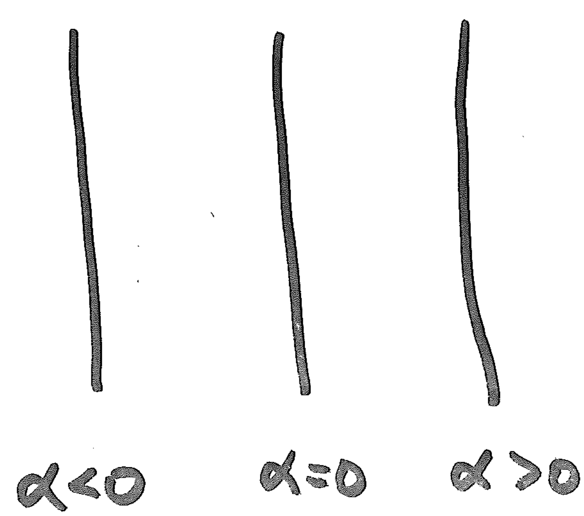
  $g_\alpha(y) > 0$ if $y > 0$

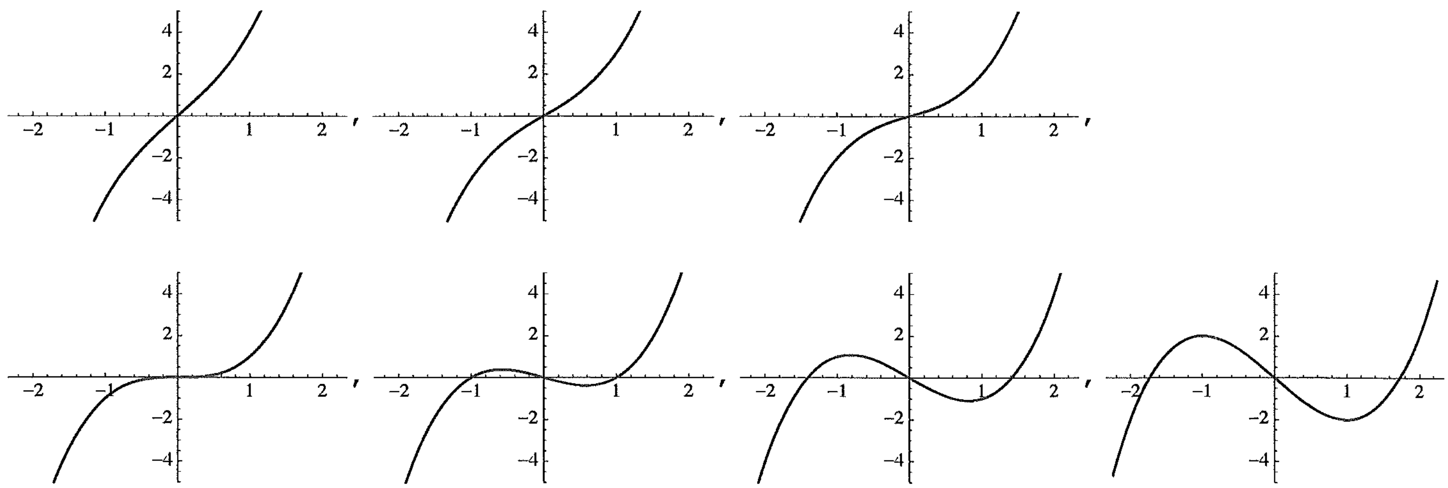
  $g_\alpha(y) < 0$ if $y < 0$

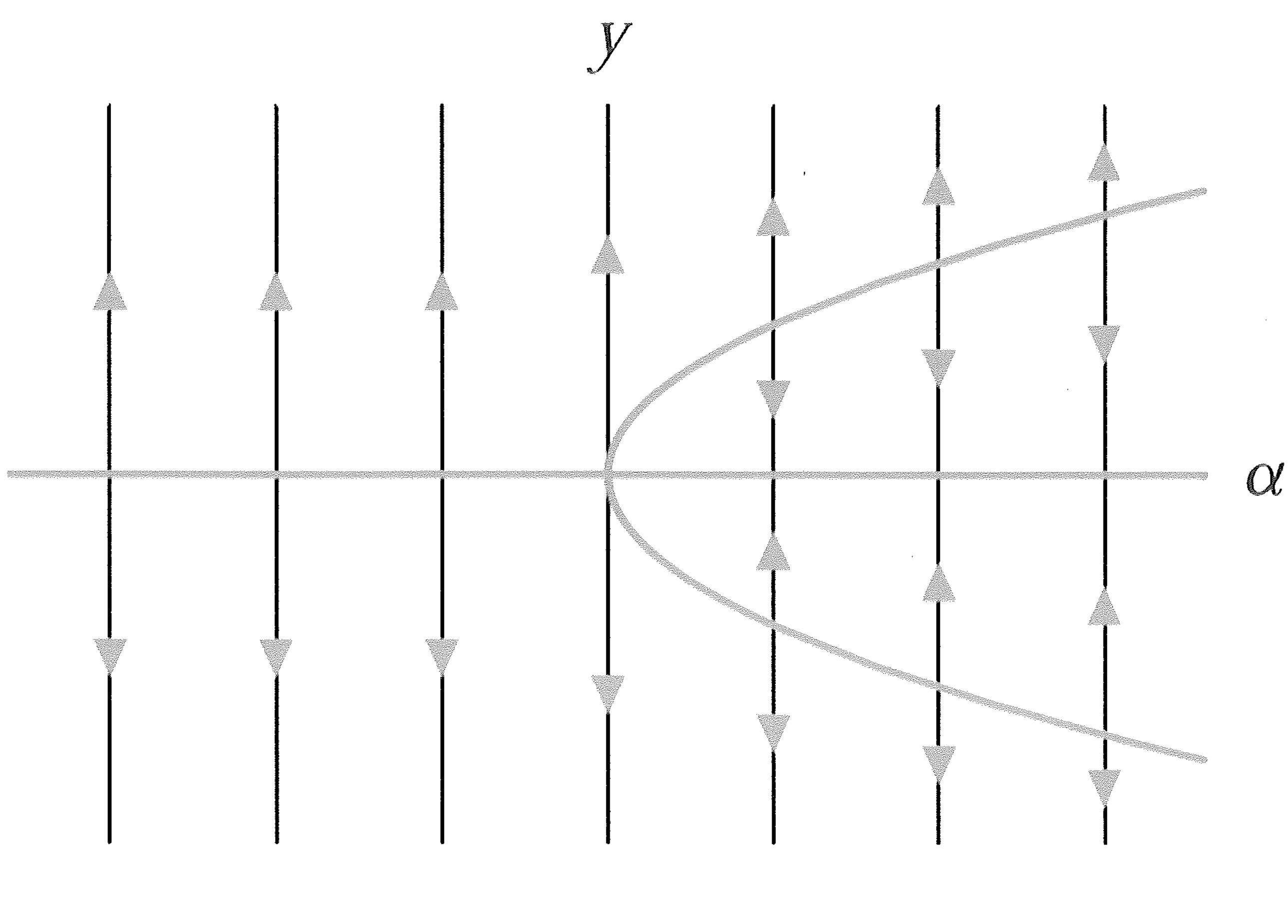
  $y = 0$ is only equilibrium point

- $\alpha > 0$ $\quad g_\alpha(y) = y(y^2 - \alpha) = 0$ when $y = 0$ or $y = \pm\sqrt{\alpha}$









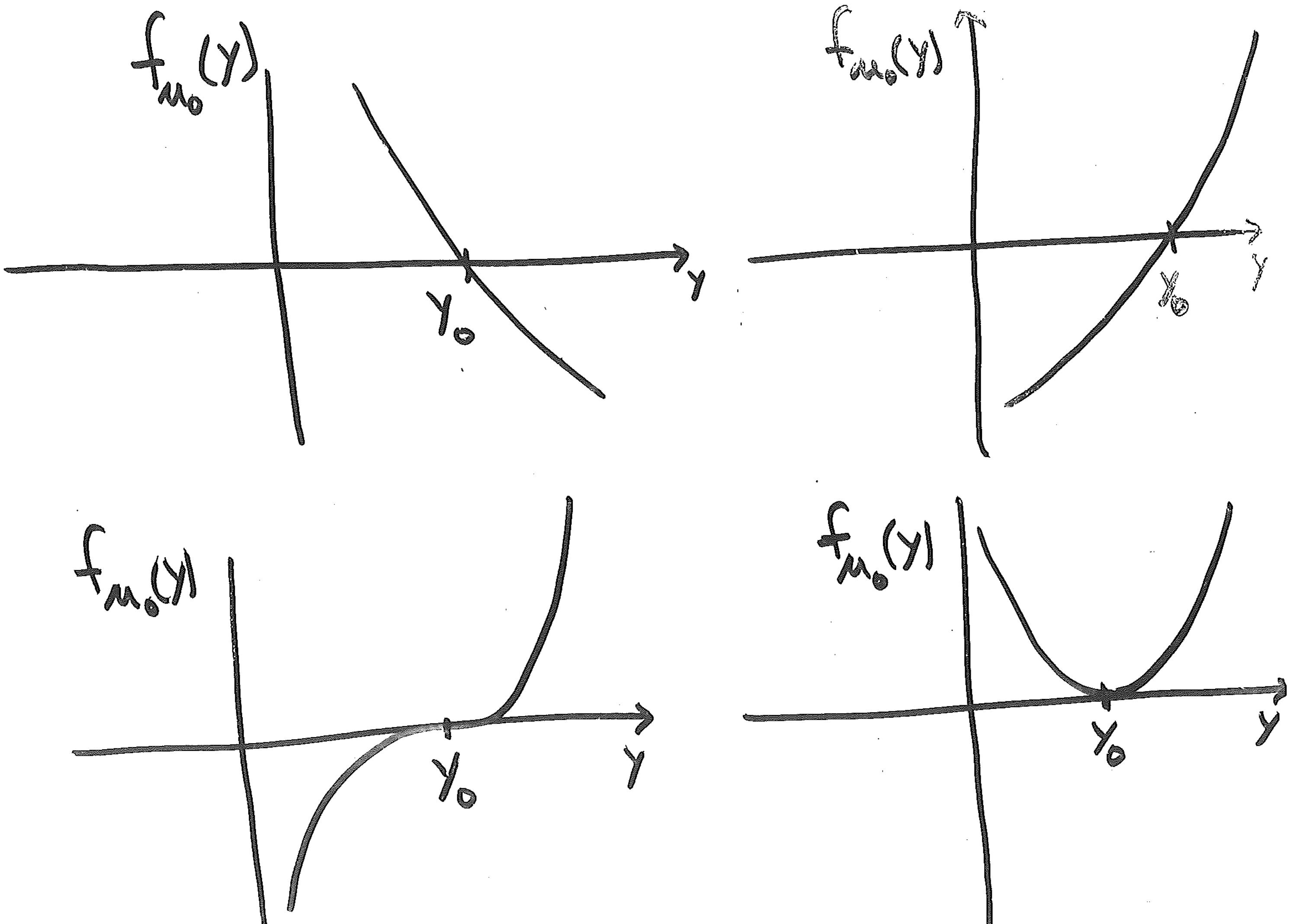
y
α

How to find bifurcation values

If $\dfrac{dy}{dt} = f_{\mu}(y)$

We are looking for situations where

$$f_{\mu_0}(y_0) = 0 \quad \text{and} \quad f'_{\mu_0}(y_0) = 0$$



Example 3

$$\frac{dy}{dt} = y(1-y)^2 + \mu = f_\mu(y)$$

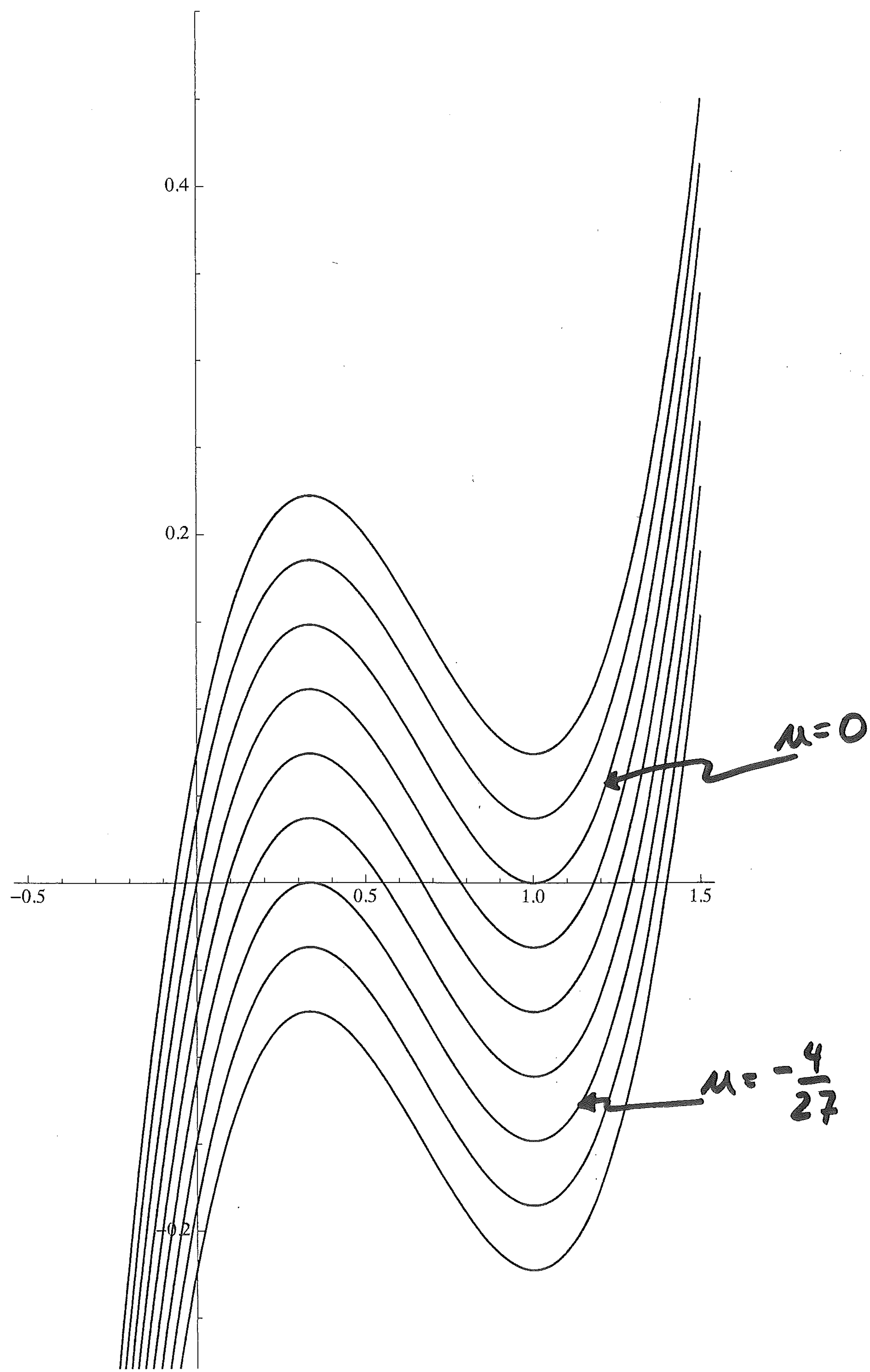
Looking for y and μ values so that

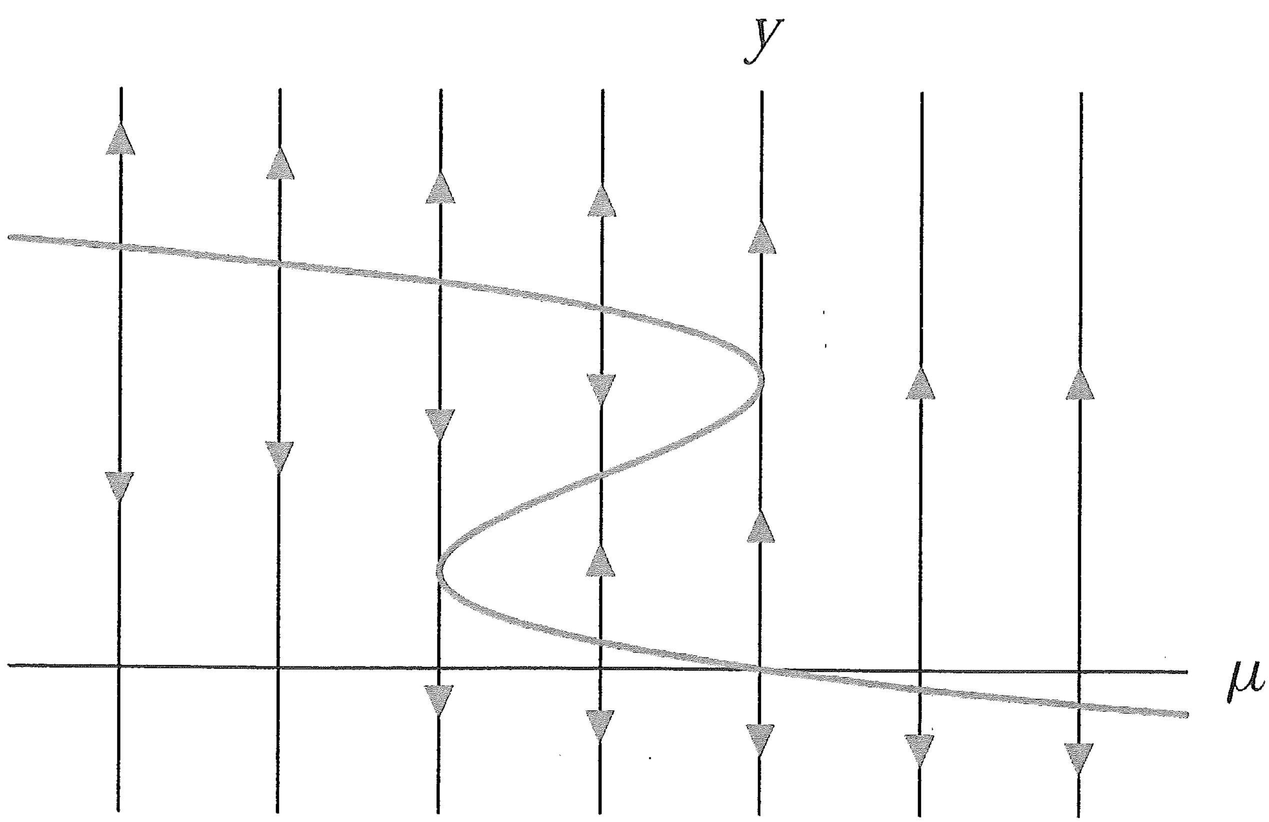
$$f_{\mu_0}(y_0) = 0 \quad \text{and} \quad f'_{\mu_0}(y_0) = 0$$

- First find y values where $f'_\mu(y) = 0$

$$f'_\mu(y) = 1(1-y)^2 - 2y(1-y) + 0$$

- Find μ so that $f_\mu(y_0) = 0$


0.4
0.2
-0.2
-0.5
0.5
1.0
1.5
$u=0$
$u=-\frac{4}{27}$


y
μ

Application - Sustainability

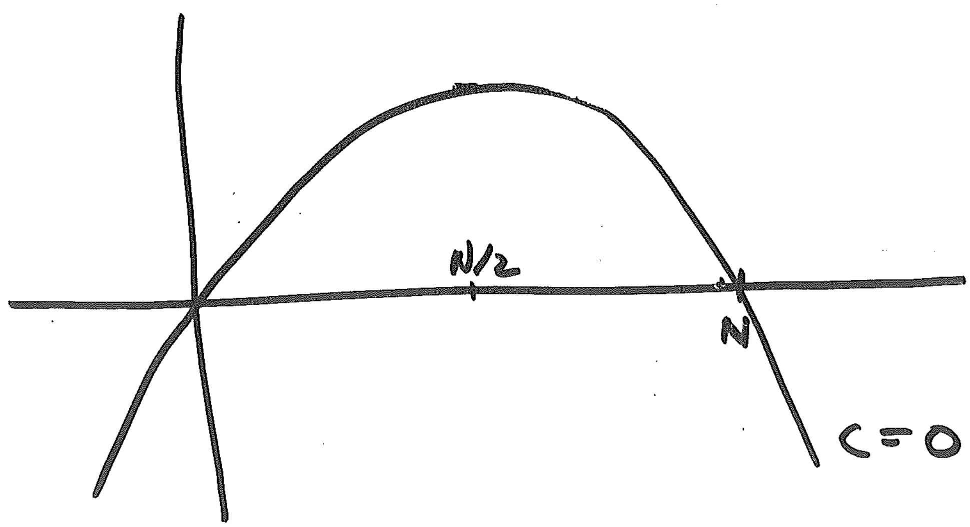
Suppose that we model a population of fish with the logistic model

$$\frac{dP}{dt} = KP\left(1 - \frac{P}{N}\right)$$

Suppose that fishing removes a constant number of fish each year which we call C. The modified model for fish population is

$$\frac{dP}{dt} = KP\left(1 - \frac{P}{N}\right) - C$$

$$= -\frac{K}{N}P^2 + KP - C$$



How to find the bifurcation value c

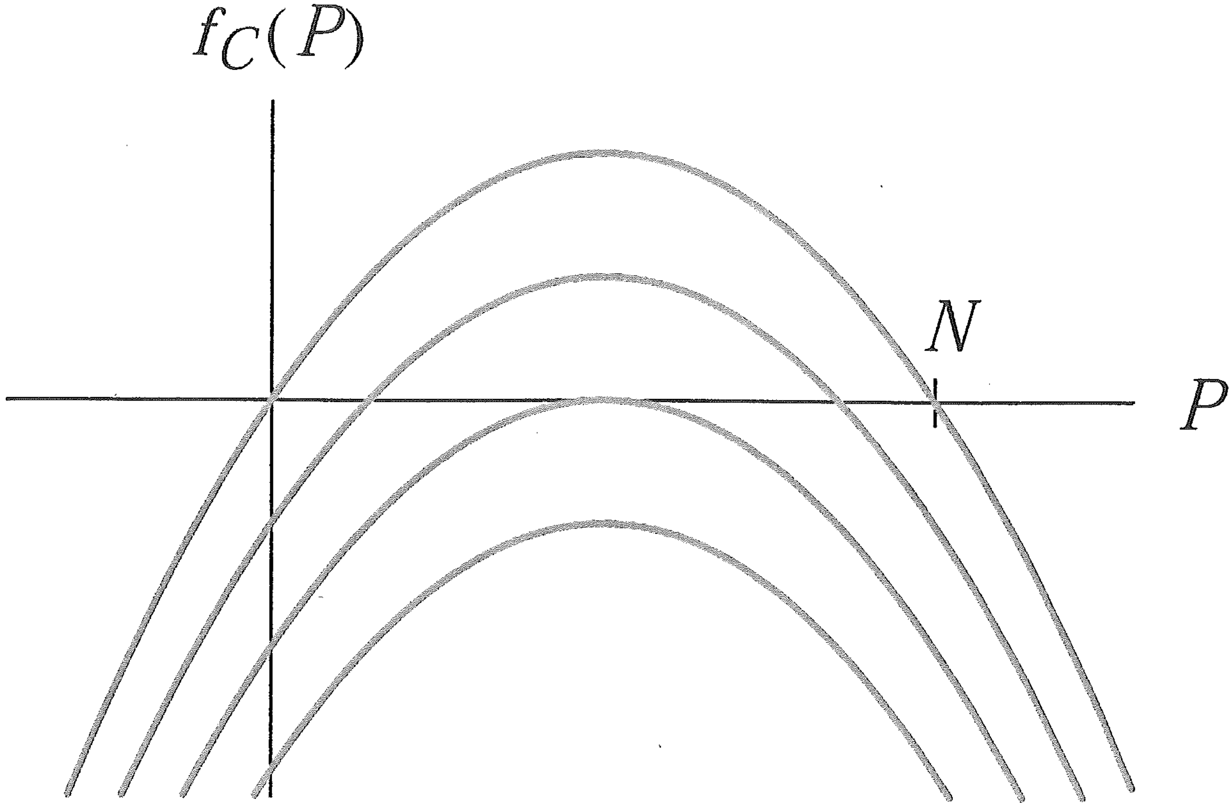
for $$\frac{dP}{dt} = KP\left(1-\frac{P}{N}\right) - c = f_c(P)$$

<u>Ask</u>: $f_c'(P) = 0$ and $f_c(P) = 0$

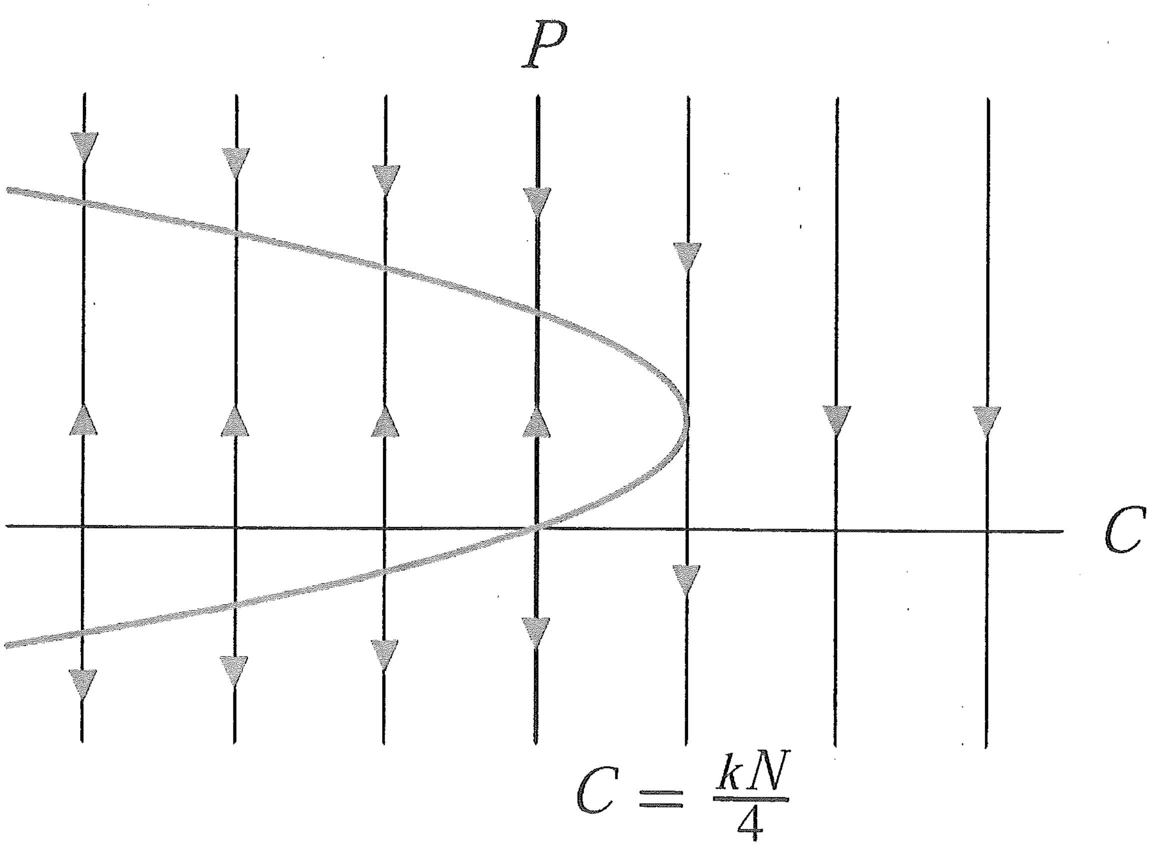
$$f_c(P) = -\frac{K}{N}P^2 + KP - c$$

$$f_c'(P) = -\frac{2K}{N}P + K$$

$$f_c'(P) = 0$$



$f_C(P)$
N
P



P
C
$C = \frac{kN}{4}$