Sec 1.6 Equilibria and the Phase Line

A differential equation of the ~~form~~ form

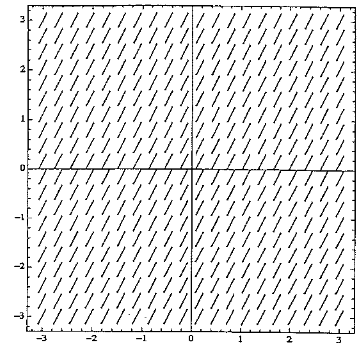
$$\frac{dy}{dt} = f(y) \qquad \text{independent of } t$$

is called autonomous. The slope fields of autonomous differential equations can be represented by 1-dimensional diagrams called phase lines.

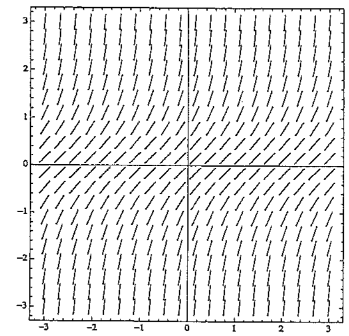
Goals:
1. Given an autonomous differential equation construct the phase line
2. Given a phase line sketch the possible solutions
3. Understand the role of equilibrium points in the interpretation of phase lines
4. Classify equilibrium points as sinks, sources or nodes.

Sec 1.6 Phase Lines

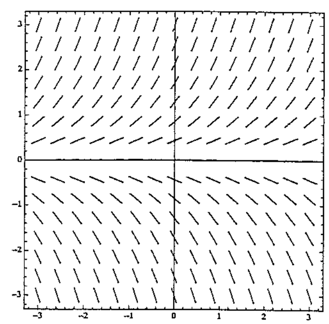
Example 1: $\frac{dy}{dt} = 2$



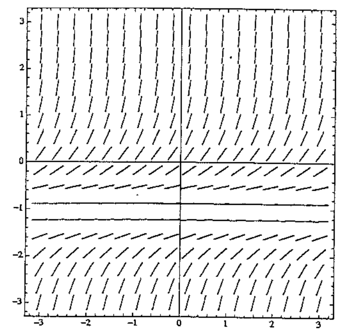
Example 1b: $\frac{dy}{dt} = 1 + y^2$



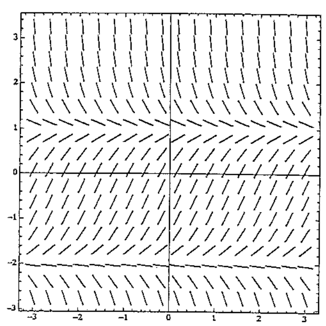
Example 2: $\frac{dy}{dt} = y$



Example 3: $\frac{dy}{dt} = (1+y)^2$



Example 4 : $\frac{dy}{dt} = (1-y)(y+2)$

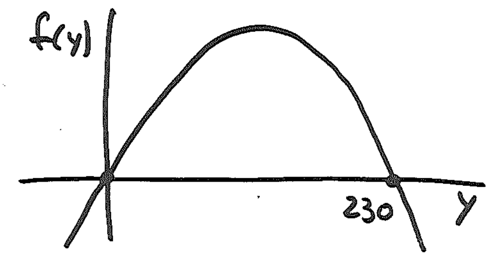


Drawing a phase line given a differential equation $\frac{dy}{dt} = f(y)$

<u>Example</u> $\frac{dy}{dt} = 0.4\,y\left(1 - \frac{y}{230}\right)$ (HW #3 Sec 1.1)

1.) Equilibrium values $y=0$, $y=230$

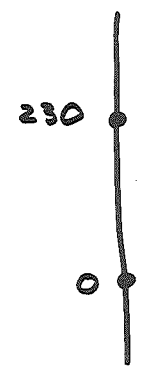
2.)



$\frac{dy}{dt} > 0$ when $0 < y < 230$

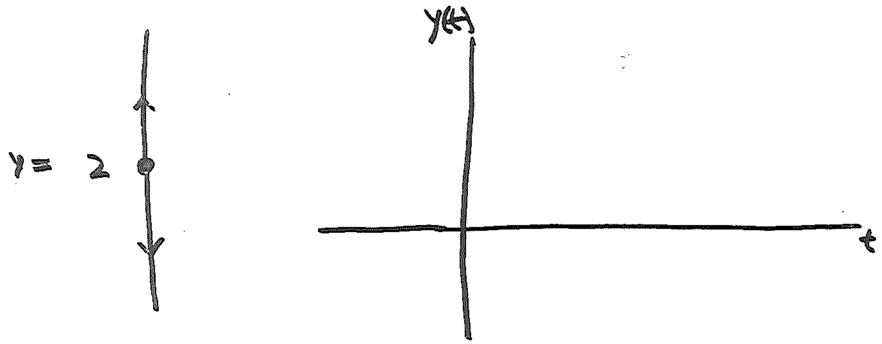
$\frac{dy}{dt} < 0$ when $y < 0$, $y > 230$

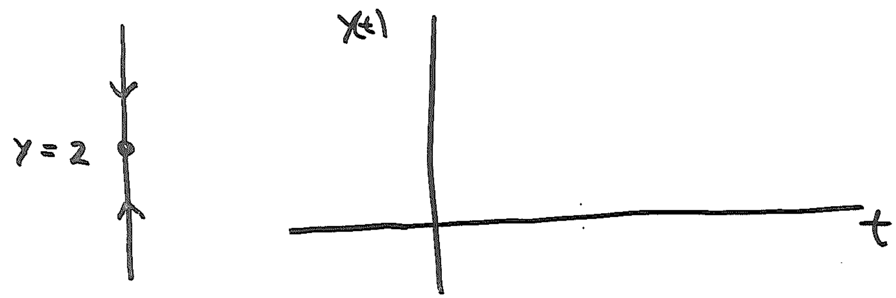
3.) Phase line

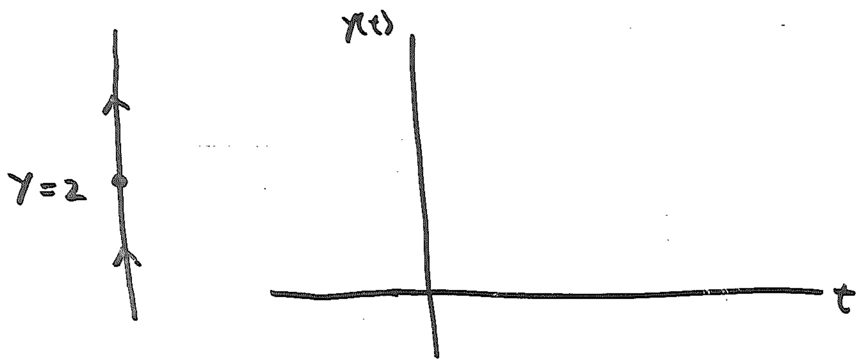


Sketching solutions given a phase line

Examples

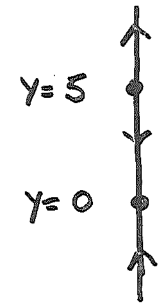


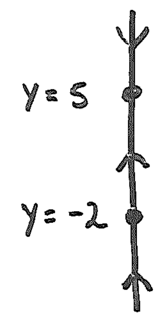


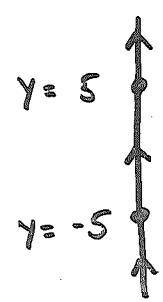


Sketching Solutions given a phase line

more examples







General guidelines when sketching solutions from phase lines (P. 78, 79)

Suppose $\frac{dy}{dt} = f(y)$ and $y(t)$ is a solution where $f(y)$ is continuously differentiable for all y.

- If $f(y(0)) = 0$, then $y(0)$ is an equilibrium point and $y(t) = 0$ for all t
- If $f(y(0)) > 0$, then $y(t)$ is increasing for all t and either $y(t) \to \infty$ as t increase or $y(t)$ tends to the first equilibrium point larger than $y(0)$
- If $f(y(0)) < 0$, then $y(t)$ is decreasing for all t and either $y(t) \to -\infty$ as t increases or $y(t)$ tends to the first equilibrium point smaller than $y(0)$.

Caution #1 Not all solutions exist for all time

Example: $\frac{dy}{dt} = (1+y)^2$

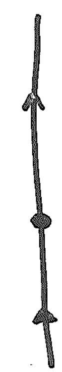
Separation of variables $\Rightarrow$ $y(t) = -1 - \frac{1}{t+C}$

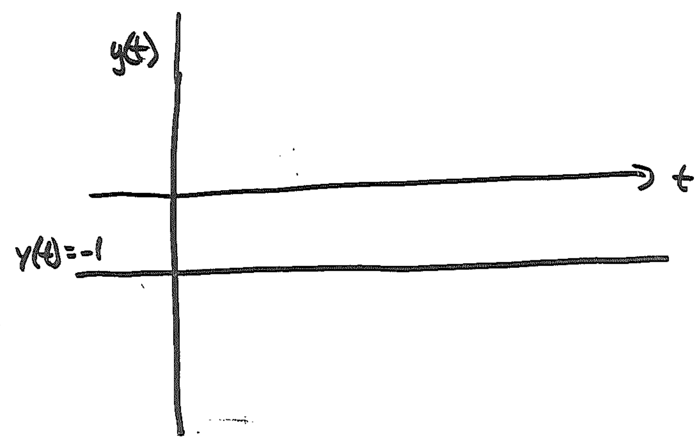
Consider initial condition $y(0)=0$

then $y(0) = 0 = -1 - \frac{1}{0+C}$ $\quad C = -1$

$y(t) = -1 - \frac{1}{t-1}$ $\quad$ vertical asymptote at $t=1$

$y = -1$





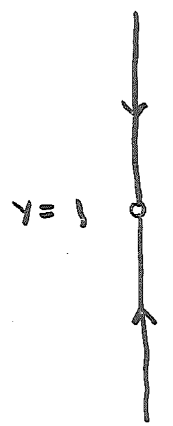
Caution #2 The phase line has a hole

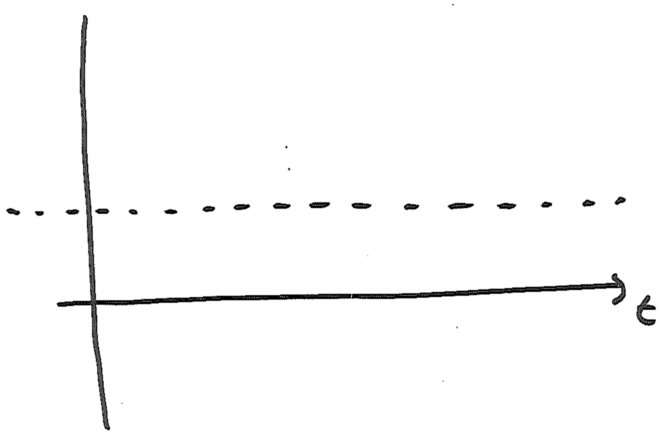
Example: $\frac{dy}{dt} = \frac{1}{1-y}$

when $y = 1$, $\frac{dy}{dt}$ is undefined

$y > 1$ $\quad \frac{dy}{dt} < 0$

$y < 0$ $\quad \frac{dy}{dt} > 0$

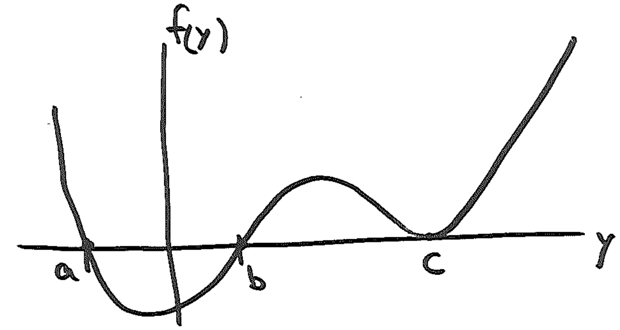


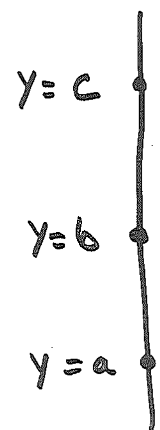


Linearization Theorem

Let $\frac{dy}{dt} = f(y)$

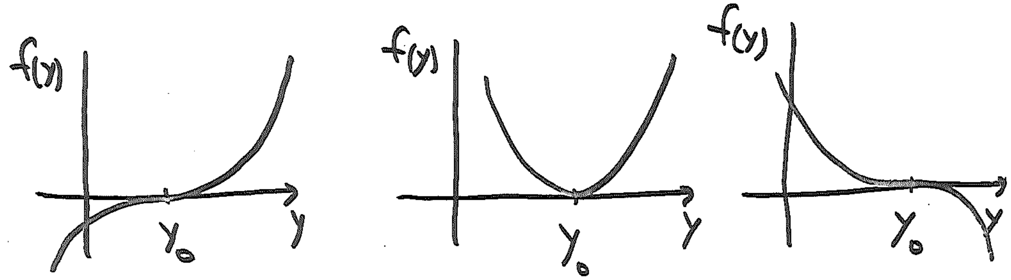
Suppose the graph of f looks like





Three cases where $f'(y)=0$

If $\frac{dy}{dt} = f(y)$ and $f'(y_0)=0$ then we could have one of the following:



Linearization Theorem Suppose y_0 is an equilibrium point of $\frac{dy}{dt} = f(y)$ where f is a continuously differentiable function. Then,

- If $f'(y_0) < 0$, then y_0 is a sink
- If $f'(y_0) > 0$, then y_0 is a source
- If $f'(y_0) = 0$, then we need additional information to determine the type of y_0

Example using Linearization

Given $\frac{dy}{dt} = h(y) = y(\cos(y^5+2y) - 27\pi y^4)$

We know that $y=0$ is an equilibrium point. Is it a source or a sink?